On Pricing of the Up-and-Out Call–A Boundary Integral Method Approach

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Abstract

The payoffs of the barrier options depend on the time path of the underlying price as opposed to just the price at expiry. It implies that both the boundary conditions and the initial condition are imposed on the Black-Scholes partial differential equation. Therefore, the valuation of the barrier options is a boundary value problem. Since the Black-Scholes equation can be converted into a homogeneous linear equation, the boundary integral method will be the most efficient numerical method to calculate the numerical solution for the barrier options. This paper demonstrates the valuation of an up-and-out call by applying the boundary integral method, and explains its risk characteristics.

Keywords: Boundary value problem; the boundary Integral method; the up-and-out call

1. Introduction

Presently, barrier options are the most heavily traded derivative products on the market. Though most common in foreign exchange, they are used extensively to manage risk related to equity, interest rates, and commodities. The main reason for the prevalence of barrier options is due to their limited risk exposure feature. The investors, particularly, can take advantage of their view about the path that the underlying asset will take through the contract period. The limited risk exposure characteristic also makes the barrier option a cheaper alternative than vanilla options with similar attributes. On the Taiwan Stock Exchange (T.S.E.), equity-based up-and-out call (henceforth, UOC) is among the most popular listed option products. Since Polaris Securities issued the first exchange-traded UOC on United Microelectronics Corporation in November 1999, the issuances of UOC have been very popular among the major brokerage houses in Taiwan. Currently, Yuanta Core Pacific Securities has become the most prominent issuer of UOC on Taiwan’s stock market. The cost effectiveness and flexibility of managing risk exposure promote UOC as one of the most actively traded derivatives on Taiwan’s stock market.

The payoff of a UOC depends on the path of the underlying price through its contract period. For a continuous UOC, only if the underlying asset price breaches the barrier before maturity, the UOC will then receive an immediate pre-specified rebate. Otherwise, the UOC will behave like a vanilla call, and the holders receive the payoff of a vanilla call. On the Taiwan Stock Exchange, the knock-out barriers for a listed call are usually set 50% higher than the exercise prices, and the immediate rebates are frequently equal to the differences between the exercise price and the underlying stock’s closing price on the barrier-breaching day. Due to the knock-out features, the characteristics of these UOCs are different from the regular option upon the impact of volatility, time to maturity, and the underlying asset price.


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Dirichlet lattice to value option with non-linear barriers, and claim that it converges considerably faster than a plain lattice scheme.

The risk-neutral expectation approach by Cox and Ross (1976) is one of the most widely applied algorithms for pricing a wide range of European option products. (e.g. Carr (1995), Rubinstein, and Reiner (1991), Gray and Whaley (1999)). However, the analytical form solutions generally suffer from some of the following potential constraints. First, although hitting time theorem can be used to price barrier option with more complex features, the algorithm is applicable under the limit that the boundary constraint has to be a constant or exponential function of time. Second, the high-dimensional numerical integration expressed in the expectation form can hardly be computed practically if the dimensions are greater than four or five. (See Ahn, Figlewski, and Gao (1999), Reiner (2000)). Third, if the transition or hitting density function does not exist, as in the case of exotic options with more complex features, this approach cannot be applied. Finally, it cannot be readily adapted to price American options.

Monte Carlo methods by Boyle (1977) and Boyle, Broadie, and Glasserman (1999) are another form of risk-neutral approach. Under the assumption that the market is composed only of the risk-neutral investors (Cox and Ross (1976), the equilibrium rate of return of all assets will be equal to the risk-free rate of return, and the theoretical value of the option will be its discounted expected terminal payoff. Experiments can be conducted to estimate an option’s expected terminal payoff and its theoretical value. Although the simulation approach can be applied to overcome a complex boundary constraint problem encountered in risk-neutral expectation approach, it still suffers from a slow convergence and simulation bias caused by the possibility of boundary constraints being hit between time steps. Therefore, it may take a rather long time to acquire a desired accurate numerical solution. This method is generally used only when other numerical techniques are unable to value the complex contingent claim.

The binomial or trinomial lattice methods by Cox, Ross, and Rubinstein (CRR) (1979), Rendleman and Bartter (1979), and Boyle (1986) are among the most popular numerical techniques to value a variety of options. The valuation algorithm proceeds backward through time with new nodes being calculated as the present value of the quasi expected value of the preceding two nodes. Each of those computed values in the new nodes are checked for the prospect of triggering complex provisions. If the complex provisions mandate the specific payoff, the new value will then replace the original computed value at that node. The iterative algorithm will come forth backward through time until the current value is finally derived. When the lattice model is chosen, you are implicitly assuming that stock prices move discretely. Unless you use a lattice with an infinitesimal mesh, the distribution error will lead to a sluggish convergence. For vanilla options, the distribution error occurs only when the underlying asset approaches the strike price at maturity; therefore the lattice models converge fairly rapidly as the levels of the tree increase. However, the UOC has a continuous or discrete boundary, and the distribution error may occur at many nodes. Thus, it is more difficult to value UOC numerically either on a binomial or trinomial tree.

Boyle and Lau (1994) suggested an algorithm to arrange the CRR tree such that the actual barrier will lie almost exactly on a layer of horizontal nodes in the tree, thus, the convergence will improve significantly. Nevertheless, it will be very difficult for the modified CRR to handle the changing barrier problem. The trinomial method by Ritchken (1995) has an extra freedom that will allow the barrier to lie exactly on a horizontal set of nodes, but it cannot provide hedge parameters for the barrier options. The adaptive mesh method by Figlewski and Gao (1999) can improve the distribution error by making the lattice finer, but halving the stock price node spacing will generally increase the number of time steps by a factor of four (see Zvan, Vetzal, and Forsyth (2000)). Therefore, a doubling of accuracy in numerical solution may ultimately increase the computation time by a factor of four, and that may induce slow convergence. In general, the lattice model will still require very large time steps if the underlying asset price is close to the barrier.

Due to the omission of higher order derivatives in the Taylor expansion, a finite difference scheme is subject to truncation error. The approximation of continuous log-normal distribution by a discrete grid will also lead to a discretization bias and a sluggish convergence for a finite difference technique. To reduce discretization bias in a finite difference method, it is theoretically possible to make the grid finer so that the discrete grid is a better approximation to the continuous distribution. Nevertheless, when the grids for a variable become finer, the accumulated errors may in fact increase, and lead to unpredictable convergence properties. In brief, finite difference schemes are a domain type numerical analysis. Although it can be universally applied to general-type PDE, truncation errors and discretization bias will reduce its accuracy.

The integral method is by far the most efficient and accurate numerical method for calculating the numerical value of a homogeneous linear PDE. In fact, the path-breaking paper by Black-Scholes (1973) also adopted the integral method in determining the first explicit general equilibrium solution for vanilla calls and puts. Our reason for choosing this approach is fourfold: (i) it is a simple, well-developed, efficient numerical solution approach being applied in many fields; (ii) the Green’s function in integral representation can be interpreted as the probability density function of the contingent claim, thus, it can be easily linked to the risk-neutral expectation approach; (iii) the integral method can be easily expanded into a boundary
element method to calculate the numerical solution of a more complex barrier option. (iv) it is extremely efficient to handle an option with discrete features, (v) the integral representation is equivalent to an analytical solution, and its accuracy and convergence is naturally superior to the lattice model.

The remainder of this paper is structured as follows. Section 2 presents a detailed discussion of the boundary integral method, including the transformation of the Black-Scholes (1973) PDE into the Heat Equation, the explanation of Green’s function, and integral representation. In section 3 we demonstrate the application of the integral method into the pricing of the continuous UOC. Section 4 provides an illustrative valuation example by using a UOC, and examines its risk characteristics. Section 5 summarizes and concludes.

2. Integral Method

2.1 The Black-Scholes PDE and Boundary Constraints

Under the assumption of the standard lognormal Black-Scholes (1973) setting, the famous Black-Scholes (1973) PDE for an option with the underlying asset following geometric Brownian motion can be written as:

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r_f S \frac{\partial C}{\partial t} + r S C = C_{tt} \]  

(1)

where \( S \) is the price of the underlying, \( \sigma \) is its volatility, \( t \) is the current time, \( r_f \) is the risk-free interest rate and \( C(S,t) \) denotes the price of the derivative security. As long as a riskless hedge can be formed between the option and the underlying asset, Eq. (1) will be the universal PDE suitable to all types of options. The option’s terminal payoff or other constraints due to the contract specification will form the boundary constraints of the option. For illustrative purposes, this paper will only discuss the boundary constraints of a vanilla call and the UOC.

For a vanilla call, its boundary condition is determined by its terminal payoff, and will be:

\[ C(S,T) = \begin{cases} 0 & \text{if } S(T) \leq K \\ S - K & \text{if } S(T) > K \end{cases} \]

(2)

Where \( T \) is the time of expiration, and \( K \) denotes the exercise price.

For a UOC, should the price of the underlying asset reach a pre-specified barrier, \( H \), the contract will then receive an immediate rebate \( R_b \); otherwise the call behaves as a vanilla call. This leads to the following boundary conditions:

\[ C(S,T) = \begin{cases} 0 & \text{if } S(T) \leq K \\ S(T) - K & \text{if } K < S(T) \leq H \\ C(S(t), t) = R_b & \text{if } S(t) \geq H \end{cases} \]

(3)

\[ C(S(t), t) = R_b & \text{if } S(t) \geq H \]

(4)

2.2. The Transformation of Variables and the Heat Equation

The Black-Scholes (1973) PDE that depicts the price behavior of the derivative security is a forward-type, and, in general, will result in an ill-post problem. If the time to the maturity of the contract is defined as \( \tau \), then \( \tau = T - t \). Thus, we change the forward-type PDE in Eq. (1) into a backward-type PDE, which is then well-posed. The backward-type PDE is as follows:

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial C}{\partial S^2} + r_f S \frac{\partial C}{\partial \tau} + r S C - C_{\tau} = 0 \]

(5)

We make the following transformation in Eq. (5):

\[ \mu = r_f - \frac{1}{2} \sigma^2 \]

(6)

\[ x = \ln S + \mu \tau \]

(7)

\[ u(x,S,\tau) = e^{-\mu \tau} C(e^{-\mu \tau}, \tau) \]

(8)

The transformation of variables has some economic meaning. Recalling that we are valuing the option at current time \( t \) but the payoff is received at the expiration date \( T \). Therefore, Eq. (8) intends to change from the present value to a future value term. Eq. (7) is a translation of the coordinate system, which makes the new variable \( x \) following the Arithmetic Brownian Motion instead of the Geometric Brownian Motion. Eq. (6) means the instantaneous drift of the Arithmetic Brownian Motion is equal to \( \mu \). It can also be interpreted as the equilibrium rate of return of the underlying asset being equal to the risk-free rate of return under a risk-neutral environment. After the transformation, the backward-type Black-Scholes (1973) PDE can then be transformed into a simpler form, and its coefficient becomes constant, independent of the underlying asset price \( S \). Ultimately, we get the Heat Equation with a constant diffusion coefficient as follows:

\[ \frac{\sigma^2}{2} u_{\tau} = u_{xx} \]

(9)

By expressing the initial and boundary conditions given in Eq. (2), (3), and (4) as a function of \( u(x,\tau) \), the boundary conditions are then:

\[ u(x,0) = \begin{cases} 0 & \text{if } e^{-\mu \tau} \leq K \\ e^{-\mu \tau} - K & \text{if } K < e^{-\mu \tau} \leq H \end{cases} \]

(10)

\[ u(x,0) = \begin{cases} 0 & \text{if } e^{-\mu \tau} \leq K \\ e^{-\mu \tau} - K & \text{if } K < e^{-\mu \tau} \leq H \end{cases} \]

(11)

\[ u(\ln H + \mu \tau, \tau) = R_b & \text{if } e^{-\mu \tau} \geq H \]

(12)

where \( \tau > 0, -\infty < x < \infty \), the immediate rebate, \( R_b \), can either be a constant or a function of time (i.e. the rebate can change through time). The barrier \( H \) can be defined as a constant, a linear function of time, or an exponential function of time. Eq. (10) is termed as the initial condition for a vanilla call’s valuation. Eqs. (11) and (12) are, respectively, the initial condition and the boundary condition for the continuous single barrier UOC. The initial condition and the boundary condition must be imposed on the Heat Equation such that (i) there exists a unique solution for each set of data, and (ii) the unique solution depends continuously on the data.
3. Integral Method and Option Pricing

To demonstrate the universal application of the integral method, this section will explain the derivation of the Green’s function, and demonstrate how it can be applied to the valuation of the continuous UOC.

3.1 The Green’s function and integral representation

The Green’s function*** for the Heat Equation is the solution to that equation at point \( x_0 \) at time \( \tau \), and it is a fundamental solution for the Heat Equation. The fundamental solution \( G(x, \tau; x_0, \tau_0) \) for the dual equation has to satisfy the following equation:

\[
\frac{1}{2} \sigma^2 G_{xx} + G_t = \delta(x-x_0, \tau-\tau_0)
\]  
(13)

where \( \delta(x-x_0, \tau-\tau_0) \) is the Dirac delta function.

Next, we multiply \( \left( \frac{\sigma^2}{2} u_{xx} - u_\tau \right) \) from Eq. (9), which is nil, by \( G(x, \tau; x_0, \tau_0) \) and proceed with a double integral. As a consequence of Eq. (13), the integral can be expressed as:

\[
\int \left( \frac{\sigma^2}{2} u_{xx} - u_\tau \right) G dx d\tau = 0
\]  
(14)

where \( D \) is defined as the domain of integration. When the domain contains only an initial condition as in Eq. (11) and the boundary is located at infinity, the Green’s function can be expressed as follows:

\[
G(x, \tau; x_0, \tau_0) = \frac{1}{\sqrt{2\pi \sigma^2(\tau-\tau)}} \exp\left( -\frac{(x-x_0)^2}{2\sigma^2(\tau-\tau)} \right) H(\tau_0-\tau)
\]  
(15)

where \( H(\tau_0-\tau) \) is the Heaviside function with the conditional value as follows:

\[
H(\tau_0-\tau) = \begin{cases} 
1, & \text{if } \tau_0 - \tau \geq 0 \\
0, & \text{if } \tau_0 - \tau < 0 
\end{cases}
\]  
(16)

Plugging in the Green’s function in Eq. (9), verifies that it is a solution to the Heat Equation. The integration by part on Eq. (14) will give:

\[
u(x_0, \tau_0) = \int_{-\infty}^{\infty} u_0(x) G(x, 0; x_0, \tau_0) dx
\]  
(17)

That is, the solution for \( u(x_0, \tau_0) \) can be expressed in terms of summing over the Green’s function’s value at point \( u_0(x) \) at the initial time as in Eq. (17).

As mentioned in John (1982), the uniqueness of the boundary value problem implies that the Green’s function always exists, although it is not always available. As long as the Green’s function exists, the solution for the PDE can always be expressed in the form of integral, and it will be termed the integral representation. Moreover, if the Green’s function is available, the integral representation can be simplified as an explicit closed-form solution as in the Black-Scholes option pricing formula (1973). In addition, the Green’s function can also be interpreted as the probability density function of the transformed underlying asset, and its derivative with respect to the boundary is the hitting probability density function. Hence, the integral representation can accordingly be described as the integrating terminal payoff with respect to the density function as in the risk-neutral expectation approach.

For vanilla options, the implicit boundary condition is located at the negative infinity of the transformed domain (i.e. \( u(-\infty, \tau) = 0 \)), and the unique solution for PDE is only subject to the initial condition. The problem for calculating the unique solution is generally called the initial value problem. If only the initial condition is imposed as a constraint on the PDE, the Green’s function for the Heat Equation is always available. Therefore, the solution for the vanilla option can be expressed as an integral representation, and can be simplified into an explicit closed-form solution. For different types of continuous single barrier options, if the barrier constraints are either a constant or an exponential boundary (its logarithm being a linear function of time), then the Green’s function is also available for them, and a closed-form solution can be derived for it as well. In general, the integral representation is an analytical form solution. Therefore, its calculation time and convergence rate is naturally much faster than the lattice model.

3.2 Valuation of Vanilla Option

For a vanilla option, \( u_0(x) \) will be its payoff at maturity. Since only the initial condition is imposed on the PDE, the solution for this problem, \( u(x_0, \tau_0) \), can be expressed in terms of summing over the Green’s function’s value at point \( u_0(x) \) at the initial time as in Eq. (17). The Green’s function in Eq. (17) can be explained as the transition density of the final payoff \( u_0(x) \), and \( u(x_0, \tau_0) \) will be the expected terminal payoff of the vanilla option. The integral representation in Eq. (17) is an analytical solution of option value, and it can be evaluated analytically as long as the Green’s function is available.

For a vanilla call, \( u_0(x) \) will be as in Eq. (10). By applying Eqs. (8) and (17), the value of a vanilla call, \( C(S, t) \), will be the discounted risk-neutral expected payoff, as in the analytical form solution derived from the risk-neutrality expectation methodology.

\[
C(S, 0) = e^{-rT} u(x_0, \tau_0)
\]  
(18)

3.3 Valuation of Single Continuous Barrier Option

The continuous barrier provision means that a boundary condition is imposed on the Black-Scholes PDE. As
long as the boundary is a constant or exponential boundary, the Green’s function is always available, thus, the numerical solution can be expressed in the form of integral representation. In this section, we only demonstrate the application of the boundary integral method (BIM) into the valuation of the UOC. However, the BIM can easily cope with other types of single barrier options.

If a constant boundary or exponential boundary such as Eq. (12) is imposed on the Heat Equation, the Green’s function will evolve into the following term:  

$$G(x, \tau; x_0, \tau_0) = \frac{1}{\sqrt{2\pi \sigma^2 (\tau - \tau_0)}} \left\{ \exp \left[ \frac{(x_0 - x)^2}{2\sigma^2 (\tau - \tau_0)} \right] - \exp \left[ \frac{(x_0 - x + b_0)^2}{2\sigma^2 (\tau - \tau_0)} \right] \right\} H(\tau - \tau_0)$$  (19)

Where:

$$b_0 = \ln H + \mu \tau$$  (20)

$$\alpha = \frac{2\mu (b_0 - x_0)}{\sigma^2}$$  (21)

and $b_0$ is the transformed barrier price according to Eq. (7).

The Green’s function in Eq. (19) can be interpreted as p.d.f. for the transformed stock price $x$. Moreover, the Green’s function in Eq. (19) has another minus term, which can also be explained as the probability that the stock price penetrates the barrier level before the expiration date, and then returns back to the price below the barrier level at the expiration date.

When a boundary condition such as Eq. (12) is imposed on the PDE, the Green’s function is then replaced by for $$u(x_0, \tau_0) = \frac{e^{-x_0^2 - \alpha}}{\sigma \sqrt{2\pi (\tau - \tau_0)}} \int_0^\tau \left\{ \exp \left[ \frac{(x_0 - x)^2}{2\sigma^2 (\tau - \tau_0)} \right] - \exp \left[ \frac{(x_0 - x + b_0)^2}{2\sigma^2 (\tau - \tau_0)} \right] \right\} H(\tau - \tau_0) \int_{-\infty}^{b_0} u_0(x) G(x, 0; x_0, \tau_0) dx$$  (22)

where:

$$f(\tau) = u_0(b_0(\tau), \tau)$$  (23)

and $b_0(\tau)$ is the derivative of $b_0(\tau)$. Hence, $b_0(\tau)$ is equal to $\mu$.

Next, since the Green’s function is zero on the boundary, the second and third term on the preceding equation will be zero, thus, Eq. (22) can be simplified as follows:  

$$u(x_0, \tau_0) = \int_{-\infty}^{b_0} u_0(x) \int_0^\tau \left\{ \exp \left[ \frac{(x_0 - x)^2}{2\sigma^2 (\tau - \tau_0)} \right] - \exp \left[ \frac{(x_0 - x + b_0)^2}{2\sigma^2 (\tau - \tau_0)} \right] \right\} H(\tau - \tau_0) \int_{-\infty}^{b_0} u_0(x) G(x, 0; x_0, \tau_0) dx$$  (24)

where:

$$G(b_0(\tau), \tau; x_0, \tau_0) = \frac{1}{\sqrt{2\pi \sigma^2 (\tau - \tau_0)}} \left\{ \exp \left[ \frac{(x_0 - x)^2}{2\sigma^2 (\tau - \tau_0)} \right] - \exp \left[ \frac{(x_0 - x + b_0)^2}{2\sigma^2 (\tau - \tau_0)} \right] \right\} H(\tau - \tau_0)$$

Since $G(b_0(\tau), \tau; x_0, \tau_0)$ is the hitting probability density function, the first term in Eq. (24) can be interpreted as the expected payoff of the UOC when the underlying asset price passes across the barrier level before the maturity date. $G$ is the transition density function of the transformed price, thus, the second term is then the expected terminal payoff for the option when the stock price never crosses the barrier level throughout the option’s contract period. An analogy to the risk-neutral expectation approach $u(x_0, \tau_0)$ is the expected terminal payoff of the UOC. By applying Eq. (8) the theoretical values of the call option is once again represented by the discounted expected payoff, as in a risk-neutral environment.

The published model by Rubinstein and Reiner (1991) is unable to handle an option with an exponential barrier, and it is also unable to handle a time-dependent rebate. The formulas derived by Kunitomo and Ikeda (1992) only consider barrier options without a rebate. The alternative valuation scheme by German and Yor (1996) involves the numerical inversion of the Laplace Transformation, and it is not computationally efficient. The integral representation as in Eq. (24) can handle an exponential boundary with a time-dependent rebate, and any simple numerical integration method can quickly derive a highly accurate value. Since the integral representation is an analytical-form solution, our method can lead to substantial improvements in calculating time and precision.

4. A Valuation Example and Estimate of Hedging Parameters

4.1 Numerical Integration

In the integral representation (24), the value of $u(x_0, \tau_0)$ can be approximated by different methods of numerical integration. The Simpson’s rule is adopted to evaluate the value for its simplicity and precise accuracy. When the function is smooth, the error term associated with Simpson’s Rule will decrease with the number of steps. Hence, the number of steps must be more delicate to reach a highly accurate numerical value.

We also need to make some adjustments to the range of integration. In theory, the probability that the absolute stock price change will be 8 standard deviations away from the current stock price is virtually zero. Thus, the integration $\int_{-\infty}^{b_0} u_0 G dx$ is approximated by: $\int_{-\infty}^{b_0} u_0 G dx$, where $b_0$
is defined as:

\[ b_i^n = \min \left\{ k_0 + 8\sigma \sqrt{T_n}, b_i \right\} \]  

(26)

The replacement of the integral range will find a highly accurate numerical solution more efficiently than will the original parameters.

4.2 Valuation Example of the UOC with a Time-dependent Rebate

When the rebate is not constant or exponentially compounding, the expected terminal value of the immediate time-varying rebate cannot be expressed in an analytical expectation form. Therefore, a risk-neutral expectation cannot be applied to value an option with a time-dependent rebate. For illustrative proposes, we will focus on a continuous UOC with a time-dependent rebate to maximize comparability with the published formulas by Rubinstein and Reiner (1991). For example, in this section the barrier and expiration date is assumed to be N.T. $150 and a half year respectively. One year is considered to consist of 365 days and the rebate will increase daily by the amount of 1/7.

Figure 1 and 2 are the plots of call premiums with respect to the stock price under the various assumptions of time to maturity. Due to the knock-out feature and the low rebate, a UOC can be a decreasing function of the stock price, and the time premium will be negative for a deep-in-the-money call. Comparisons between Figure 1 and Figure 2 show the following: (1) For an option with a longer maturity time, the negative time effect emerges even when the call is still an out-of-the-money type. For calls with a short time to maturity, the negative time effect appears on a much higher stock price level. (2) As the negative time effect begins to rise, the negative time effect will soon dominate the intrinsic value effect. Thus, the longer the time to maturity and the higher the stock price, the sooner the domination will take effect. As a result of the negative

![Figure 1](image1.png)  
Figure 1. The Valuation of a European Up-and-out Call with a Time-dependent Rebate. \( H = 150, K = 100, \sigma = 0.4, T = 0.5 \) and \( r = 0.02 \). The rebate will increase by the amount of 1/7 daily, and the barrier is monitored continuously. The time to maturity of the option is assumed to be 4 months, 5 months, and 6 months respectively.

![Figure 2](image2.png)  
Figure 2. The Valuation of a European Up-and-out Call with a Time-dependent Rebate. \( H = 150, K = 100, \sigma = 0.4, T = 0.5 \) and \( r = 0.02 \). The rebate will increase by the amount of 1/7 daily, and the barrier is monitored continuously. The time to maturity of the option is assumed to be 1 week, 2 weeks, and 1 month respectively.

![Figure 3](image3.png)  
Figure 3. The Valuation of a European Up-and-out Call with a Time-dependent Rebate. \( K = 100, \sigma = 0.4, T = 0.5 \) and \( r = 0.02 \). The rebate will increase by the amount of 1/7 daily, and the barrier is monitored continuously. The barrier \( H \) for the option is assumed to be 150, 160, and 170 respectively.

![Figure 4](image4.png)  
Figure 4. The Valuation of a European Up-and-out Call with a Time-dependent Rebate. \( H = 150, \sigma = 0.4, T = 0.5 \) and \( r = 0.02 \). The rebate will increase by the amount of 1/7 daily, and the barrier is monitored continuously. The strike price for the option is assumed to be 90, 100, and 110 respectively.
time effect, the call premium tends to be a decreasing function of time for an option with a longer time to maturity or deeper in-the-money.

Figure 3 demonstrates the effect of the barrier. For call options with a low amount of rebate, the lower the barrier, the less likely the potential payoff will be. Thus, the call option will be an increasing function of the barrier for a UOC. With a deep out-of-the-money option, the chance of hitting the barrier is low. Consequently, the calls with different barriers converge together. When the stock price begins to rise, the possibility of knocking-out appears to be significant. Hence, the barrier effect becomes emergent. Figure 4 shows the impact of the strike price upon the valuation of the UOC. Because the strike price implies the cost paid for the underlying asset, in accordance with the traditional point of view, the UOC is always a monotonically increasing function of the strike price.

Figure 5. The Valuation of a European Up-and-out Call with a Time-dependent Rebate. \( K = 100, \ H = 150, \ T = 0.5 \) and \( r = 0.02 \). The rebate will increase by the amount of 1/7 daily, and the barrier is monitored continuously. The volatility \( \sigma \) for the option is assumed to be 0.2, 0.4, and 0.6 respectively.

Figure 6. The Valuation of a European Up-and-out Call with a Zero Rebate. \( K = 100, \ H = 150, \ T = 0.5, \ \sigma = 0.4, \) and \( r = 0.02 \). The barrier is monitored continuously, and the time to maturity of the option is assumed to be 4 months, 5 months, and 6 months respectively.

In contrast to the standard option, high volatility may have a negative impact upon the value of a UOC. Due to the low amount of rebate, volatility enlargement will increase the likelihood of breaching the barrier, thus the option may end up almost worthless. Figure 5 illustrates the volatility effect. Analogous to the previous analysis, a longer time to maturity may cause a negative time effect, and an augmentation in the volatility may correspondingly reduce the option value. Because both high volatility and lengthier time to maturity may cause a negative time effect, and an augmentation in the volatility may correspondingly reduce the option value. Because both high volatility and lengthier time to maturity magnify the dispersion of underlying asset price change, they will tend to reinforce each other. Hence, it is interesting to note that a UOC with no rebate will theoretically converge to a nil value on the condition that it has either infinite maturity or unbounded volatility.

The amount of rebate plays a very crucial role on the valuation of a UOC. As the amount of rebate increases, the UOC will resemble a vanilla call in many aspects. Nonetheless, for a zero-rebate UOC, its value and major attributes will dramatically deviate from that of the vanilla call. Figure 6 and 7 depict the value of a UOC under different assumptions of rebate amount. For a UOC with a zero rebate as in Figure 6, its major attributes are highly similar to the UOC with a time-dependent rebate as discussed in the preceding section. However, when the rebate is set equal to the difference between the barrier and strike price, as in Taiwan’s call warrant market, this call will always be a monotonically increasing concave function of stock price, and a negative time effect will begin to disappear.

4.3. Estimation of Hedging Parameters

The response of the UOC to changes in its argument values is quite straightforward and conceptually simple. By applying the chain rule, the deviation of a UOC in respect to the stock price is equal to the derivative of the call with respect to the transformed price, multiplied by the derivative of the transformed price with respect to the stock price. (i.e. the hedging parameter), which can be written as:
\[ \Delta = \frac{\partial}{\partial S} C(S, t) = \frac{\partial}{\partial x} C(S, t) \frac{dx}{dS} = \left( \frac{\partial}{\partial x} u \right) e^{-r/S} = u \frac{e^{-r/S}}{S} \quad (27) \]

using the standard central difference formula, \( u_x \) is given by:

\[ \frac{\partial}{\partial x} u = \frac{u(x + \varepsilon, t) - u(x - \varepsilon, t)}{2\varepsilon} \quad (28) \]

To reconcile the need for differential accuracy and avoidance of an escalating calculation error, the \( \varepsilon \) is set equal to \( 10^{-2} \).

The estimation of the call’s sensitivity with respect to other variables is equally straightforward.

\[ \theta = \frac{\partial}{\partial t} C = -\frac{\partial}{\partial t} (ue^{-r/S}) = u_t e^{-r/S} + u e^{-r/S} \frac{\partial}{\partial t} \quad (29) \]

\[ \frac{\partial}{\partial t} u = \frac{u_t(x, t + \varepsilon) - u_t(x, t - \varepsilon)}{2\varepsilon} \quad (30) \]

\[ \gamma = \frac{\partial}{\partial \sigma} C(S, t, \sigma) = \frac{C(S, t, \sigma + \varepsilon) - C(S, t, \sigma - \varepsilon)}{2\varepsilon} \quad (31) \]

Figure 8 illustrates the sensitivity of the UOC valuation with respect to a changing stock price. Analogous to the vanilla call, the delta of the UOC is concave and monotonically increases when it is a deep out-of-the-money option. However, when the call gradually turns into an in-the-money option, the possibility of breaching the barrier increases. Thus, its delta becomes concave and monotonically decreasing. Figure 9 demonstrates the impact of the time to maturity upon the price of a UOC. Initially, the call value is an increasing function of its time to maturity. As the time to maturity of the option increases, the chance of hitting the barrier and receiving nil also increases. Therefore, the time effect becomes negative.

Figure 10 explains the relationship between the value of an out-of-the-money UOC and its volatility. Similar to the previous analysis, volatility enlargement may trigger the knock-out feature. Therefore, an increase in the volatility may actually decrease the value of a UOC.

5. Conclusions

The famous Black-Scholes (1973) PDE is a linear parabolic equation, and the integral method is by far considered to be the most efficient scheme to calculate the precise numerical solution for a linear PDE. When the forward-type PDE is changed into a backward-type PDE, the terminal payoff of the option is then the initial condition of the PDE, and the provisions specified in the option contract determines the boundary condition for the PDE. Since the Black-Scholes equation can be converted into the Heat equation, its Green’s function is always available under the constraint that the boundary is a linear function of time at the transferred domain. Therefore, the solution for the barrier option with a constant or exponential barrier can always be expressed in terms of the integral representation, and a highly accurate numerical solution can be derived through the numerical integration method. Even when a very complex feature is imposed on the option contract such that the Green’s function exists but is not available, the boundary element method (BEM) can still be applied to calculate the highly precise numerical value for the complex option.
In this paper, we only demonstrate the effectiveness of the integral method by using the UOC example, however it can be extended to value a partial barrier option, discrete barrier options, the American option, and reset options. In comparison to the lattice or finite difference method, the integral method can reach all key prices of the underlying asset in just one time step, and it can handle both stationary and mobile boundaries. The valuation model in our paper will lead to significant improvements in both accuracy and flexibility. Its readiness to expand into BEM allows the integral method to be a powerful approach to price options with complex features.

References