Long-Term Trend Analysis of Online Trading  
--A Stochastic Order Switching Model  

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Abstract

Online brokerages are replacing brokers and telephones with computers and codes, and compete intensely for investors. The investment costs for setting up an online service are far lower than starting a traditional full-service brokerage. Attracted by the low commissions and high convenience of online trading, there has been an explosion in online trading that is likely to continue in the next decade. There are many advantages and disadvantages to online trading. In this research, we study the long-term trend of investors’ orders submitted to two types of brokerages: e- and non-e-brokerages in the stock market. To understand how investors choose trading channels, we identify five important factors that affect the investors’ choice of brokerages. Since some factors are qualitative, we develop linear formulas to convert multiple factors and imbedded multiple attributes into scalars to measure investors’ overall preferences of brokerages. Based on the investors’ preference measures of brokerages, a stochastic process called the order-switching model is then developed to study the impact of investors’ preferences on the number of orders submitted to each type of brokerage in the stock market. Both analytical and empirical results are derived and provide many insightful observations.

Keywords: Online trading; Multi-attribute utility; Stochastic process; E-brokerages

1. Introduction

In recent years, there has been an explosion in online trading that is likely to continue. On-line brokerage firms like E-Trade in the U.S. are among the most active and successful financial service organizations. The number of do-it-yourself online investors has also grown at a remarkable rate since the first electronic brokerage opened its virtual doors in 1994. From 1995 through mid-2000, investors opened 12.5 million online brokerage accounts. Forrester Research, Inc. (Punishill [10]) projected “that by 2003, 9.7 million U.S. households will manage more than $3 trillion online—nearly 19 percent of total retail investment assets—in 20.4 million online accounts.”

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Online brokerages replace brokers and telephones with computers and codes, and compete intensely for investors. The investment costs for setting up an online service are far lower than starting a traditional full-service brokerage because an online brokerage needs far fewer employees, most of whom may work at one central location. According to Barber and Odean ([3]), Merrill Lynch employs 20,200 financial consultants and other investment professionals and, assuming an average commission of $210 per trade, handles 124,000 commission generating transactions per day, while E-Trade employs 2,800 people and handles approximately 283,000 transactions a day. As a result, online commissions are as low as $8, but average $15 per trade. This compares to $50-100 at a discount broker and $100-$200 at a full-service broker.

Many large banking institutions have also recognized the advantages of the Internet and begun to offer online investing. In JP Morgan, for example, about 150 employees focused on cultivating new customers for Morgan Online. After the company's merger with Chase Manhattan, employees in sales, marketing and other client acquisition roles were laid off. However, about 100 people working on the technology that drives Morgan Online kept their jobs and continued work on the site. Morgan Online distinguished itself from competitors by offering more than online stock trading. Instead, the Web site provided an integrated set of services, including account aggregation, real estate and tax planning and professional advice (http://www.internetweek.com/netresults01/net011501.htm).

Despite the rapid growth in online brokerages, traditional brokers remain important player in the stock market. This is because the new investment environment has its dark side. As pointed out: “Markets in which valuations are uncertain, investors are active and inexperienced, and money to invest is readily available are prone to speculative bubbles, which can hurt all investors”. Security is another major barrier in online trading. Many investors are holding back from online trading due the limited protection of data discretion and personal authentication provided by e-brokerages.

In this paper, we study the long-term trend of online trading orders and evaluate the trade-offs between e- and non-e-brokerages. The “e-brokerage” refers to Internet trading services, while non-e-brokerages” include full and discount service brokers. To distinguish between the two types of broker- ages, we identify five important factors that influence investors’ choice of brokerages. The five factors range quantitative factors (cost and time) to qualitative factors (convenience, service and security). To compare the two
types of brokerages, we propose measurements of the five factors. Seeking to develop a practical tool for the measurements of the factors, we follow the theory of multi-attribute utility (MAU). Based on MAU theory, we develop linear formulas to incorporate the five factors into scalar measures to represent overall investors’ preferences for the two types of brokerages. Based on the investors’ preferences, we develop a stochastic process model to describe investors’ choice between the two types of brokerages (e- or non-e-brokerage) in the stock market in the long run. Analytical results are derived from the model to explain long-term trends of submitted orders to brokerages. Finally, extensive computational experiments are performed to explain the theoretical results and derive more insights.

The remainder of this paper is organized as follows. Section 2 provides an overview of the literature of online investing. Section 3 identifies five factors that influence investors’ choice of brokerages and based on MAU theory, linear formulas are developed to incorporate the five factors and thereby represent investors’ preferences for the two types of brokerages. Section 4 proposes an order-switching model to describe investors’ changing behavior in the choice of brokerages. Section 5 presents the computational results and sensitivity analysis. Concluding remarks and extensions are discussed in Section 6.

2. Literature Review

Research on the study of electronic market and financial services is abundant over the past several years and, in general, can be divided into two streams: (i) the study of the role of IT in financial services; and (ii) the study of infrastructure changes in trading markets. While information systems and management science researchers have focused on the implication of IT to the electronic markets, economics and finance researchers have examined dynamic changes in the trading market. Thus, our review will focus on these two streams.

Varian ([15]) provided a review of academic work on electronic markets and pointed out that financial services have always been heavy users of IT and thus online brokerages have been quick to take advantage of the Internet’s potential. Karmarkar ([7]) examined the impact of new technologies on customer access and the interaction between customer and service provider. He found that access played an important competitive role in financial service. Dewan and Mendelson ([5]) used Schwab as a case study to illustrate the importance of the Internet channel and the role of IT and the Internet in the creation of value in financial services. They show that the Internet has made
its way to the core of Schwab’s business strategy. In a study by Konana, Menon and Balasubramanian ([9]), it was discussed that when the entire trading process is transparent, consumer choice related to brokerage selection is more discriminating.

When the financial markets started to change dramatically in early 1990s, researchers in finance began to examine trading mechanisms and systems. Early research compared floor trading and screen trading (Shyy and Lee [14]; and Kempf and Korn [8]). Barber and Odean ([3], [4]) studied how technological developments associated with the Internet are likely to affect investors and financial markets. They found that young men who are active traders with high incomes and a preference for investing in small growth stocks with high market risk are likely to switch to online trading. They also found that those who switched to online trading usually had strong performance prior to going online, beating the market by more than two percent annually. After going online, they traded more actively, more speculatively, and less profitably than before - lagging the market by more than three percent annually. Hosking, Kambil and Lister ([6]) pointed out that electronic commerce provides new capabilities to both investors and providers of financial services. To cope with the various changes and transitions in financial services, managers of traditional brokerages will have to rethink their segmentation models and reconstruct the value bundles offered in the electronic marketplace. Rajgopal, Kotha and Venkatachalam ([11]) found that web traffic is relevant in explaining market values and stock returns of pure Internet companies. However, they found weak associations between web traffic and sales levels and sales growth.

Although the above papers provided interesting findings and discussed the benefits and weaknesses of online trading, none of them investigated the long-term impact of online trading. Bakos ([2]) concluded that institutional rules, regulations, and monitoring functions would play a significant role in promoting online trading. He also pointed out that Internet-based electronic marketplaces would promote greater economic efficiency, and help sustain economic growth. Although the conclusion is interesting, Bakos ([2]) did not provide analytical models to illustrate their statement. In this study, we aim at developing an analytical model to examine the long-term impact of online trading. We believe when online trading becomes more popular, the organizational and technological structure of banking and brokerage systems will also change accordingly. Thus, our study will pave the way for understanding the gradual change in the stock market in the long run. Specifically we identify major factors that affect investors’ choice of trading channels and
the long-run impact of e-brokerages on investors’ order submission. In this paper, we focus our study on the modeling of investors’ choice of two types of brokerages: e- and non-e-brokerages, and illustrate the long-term trend of the orders submitted to brokerages in the stock market. To our knowledge, our research constitutes the first attempt to quantify investors’ preferences of brokerages and is the first to develop a modeling approach to verify the qualitative statements about the future of e-brokerages discussed in early literature.

3. Five Factors and Their Measurements

Hosking, Kambil and Lister ([6]) pointed out that the growth in online investing could be attributed to three factors: technological advances, changes in individual investor attitudes, and favorable market conditions. The use of the Internet provides individual investors with increased capabilities to transact. This includes improved access to information, greater convenience in trading at any time of the day, low online commissions, and no need to interact with brokers. The impressive industry growth in trading has increased entry and price competition. As Barber and Odean ([3]) pointed out, online investors behave differently from traditional stockholders. E-trade claimed its customers averaged 25 trades per year, which is much more than customers of full-service brokerages (1 to 2 times per year) or discount brokerage (4 to 6 times per year). From a psychological perspective, Rabin and Schrag ([12]) indicated that people have a cognitive bias that leads them to misinterpret new information as supporting previously held hypotheses. Their study shows that this bias may induce overconfidence in people’s behavior.

Clearly, the number of do-it-yourself investors, new or switched from traditional brokerages, has grown exponentially to take advantage of the low commissions and convenience offered by e-brokerages. To examine how and why investors choose e- vs. non-e-brokerages, we propose five major factors to illustrate the investors’ choices between e- and non-e-brokerages. We believe the five factors highlighted in the early literature have played important roles in leading investors to make choices of brokerages:

- Commission: the fixed charge by a brokerage to make a trade
- Submission time: the period from the time of initiation of an order to the time the order is submitted to a brokerage.
- Convenience: the ease of access to information and flexibilities in trade time, locations, frequencies, and facilities.
- Provided Service: professional advice, direct interaction with brokerage representatives, and handling of complaints.
• **Security**: protected information, accounts and privacy.

Recognizing the difficulty in comparing the above five factors, we propose to convert the measures of the factors from multiple values to a scalar measure by developing a multi-factor linear formula based on the multi-attribute utility (MAU) theory. The MAU theory is one of the major analytical tools associated with the field of decision analysis. A MAU analysis of alternatives explicitly identifies the measures that are used to evaluate the alternatives, and helps to identify those alternatives that perform well on a majority of these measures, with a special emphasis on the measures that are considered to be relatively more important. There are many forms of MAU functions that are theoretically valid with the multi-attribute linear utility function (LA) as the most commonly used. Here, we select linear multi-factor formulas, and define two functions as below:

\[ u_e = \sum_{i=1}^{5} w_{ie} u_{ie} \quad (1.1) \]

\[ u_n = \sum_{i=1}^{5} w_{in} u_{in} \quad (1.2) \]

The single measures, \( u_e \) and \( u_n \), represent overall investors’ preferences of e- and non-e-brokerages with \( 0 \leq u_e \leq 1 \) and \( 0 \leq u_n \leq 1 \). \( u_{ie} \) and \( u_{in} (i = 1, \ldots, 5) \) represent the preference measures of each of the five factors and are scaled from 0 to 1. \( w_{ie} \) and \( w_{in} (\forall i) \) are the weights of \( u_{ie} \) and \( u_{in} \) with \( 0 \leq w_{ie} \), \( w_{in} \leq 1 \) for all \( i \) and \( \sum_{i=1}^{5} w_{ie} = 1 \), \( \sum_{i=1}^{5} w_{in} = 1 \). Note that the subscripts of \( e \) and \( n \) represent e-brokerages and non-e-brokerages.

It is interesting to notice that the LA is rather easy to implement due to its additive format. To estimate the weights in the LA, two approaches can be adopted. The first one is called the trade-off method by which a decision maker is asked to do pair-wise comparisons between each measure \( i \) \((i = 1, 2, \ldots, I)\) with other measures and obtain a weight for each measure. This procedure also requires that the weights of all measures should sum into one. Another popular approach is called analytical hierarchy process (AHP) developed by Saaty [13]. Similarly AHP also requires pairwise comparisons but it seems more complicated in implementation.
Note in (1.1) and (1.2), \( u_{ie} \) and \( u_{in} \) are independent of \( u_{ke} \) and \( u_{kn} \) \((i \neq k)\). If \( i \)th factor, \( u_{ie} \) and \( u_{in} \), consists of multiple attributes, the measures of \( u_{ie} \) and \( u_{in} \) can also be expressed as multi-attribute linear formulas as below,

\[
\begin{align*}
\sum_{j=1}^{n_i} a_{ie_j} \beta_{ie_j} & = u_{ie} \quad (2.1) \\
\sum_{j=1}^{n_i} a_{in_j} \beta_{in_j} & = u_{in} \quad (2.2)
\end{align*}
\]

where \( \beta_{ie_j} \) and \( \beta_{in_j} \) \((j = 1, \ldots, n_i \) and \( 0 \leq \beta_{ie_j} \leq 1, 0 \leq \beta_{in_j} \leq 1)\) are the preference measures of \( j \)th attribute in \( u_{ie} \) and \( u_{in} \), \( a_{ie_j} \) and \( a_{in_j} \) are the weights of \( \beta_{ie_j} \) and \( \beta_{in_j} \). We assume \( 0 \leq a_{ie_j}, a_{in_j} \leq 1 \) and \( \sum_{j=1}^{n_i} a_{ie_j} = 1 \), \( \sum_{j=1}^{n_i} a_{in_j} = 1 \). \( n_i \) is the number of the attributes in \( u_{ie} \) and \( u_{in} \).

Before we discuss the preference measures of the five factors and their attributes, we propose two assumptions as below:

**Assumptions:**

1. We compare and evaluate investors’ preferences of two types of brokerages: e-brokerages vs. non-e-brokerages. A trade via an e-brokerage is performed on the Internet and a trade via a non-e-brokerage is made through calling or contacting an individual broker.
2. The trading activities, in general, refer to market and limit orders.

We now discuss the development of the preference measures of the five factors and their attributes. We consider one factor a time and explain how the preference measure of each factor can be generated. For simplicity, we do not distinguish between first-time and experienced investors making the choice of brokerages.

3.1 Preference Measures of Commissions, \( u_{1e} \) and \( u_{1n} \)

Following the basic supply and demand equations in economics, we assume investors’ demand for trades (\( Q \)) in the market is a function of commission as below:

\[
Q = a - b * C
\]
where \( a \) and \( b \) are parameters and \( C \) represents an average commission. Clearly, if commission decreases, the demand for trading increases at: \( \Delta Q = -b^* \Delta C \).

We may replace \( b \) by \( b_e \) and \( b_n \), and \( C \) by \( C_e \) and \( C_n \) to differentiate e- and non-e-brokerages. Then \( C_e \) and \( C_n \) represent average commissions per trade at e- and non-e-brokerages. Let \( C_{high} \) and \( C_{low} \) be the upper and lower bounds of the commissions in the stock market. We now define the preference measures of commissions at e- and non-e-brokerages as below:

\[
 u_{te} = b_e \frac{C_{high} - C_e}{C_{high} - C_{low}} \quad \text{and} \quad u_{tn} = b_n \frac{C_{high} - C_n}{C_{high} - C_{low}}
\]

In the case of \( b_e = b_n = 1 \) and \( C_e < C_n \) (the average commission at e-brokerages is smaller than that at non-e-brokerages), then \( u_{te} > u_{tn} \). This means that if the commission is the only criterion in making choice of brokerages, investors prefer e-brokerages to non-e-brokerages.

### 3.2 Preference Measures of Submission Time, \( u_{2e} \) and \( u_{2n} \)

A trade, in general, can be divided into two stages. In the first stage, investors choose a trading channel (e.g. a brokerage) and then submit their orders (either market or limit orders) to the brokerage. In the second stage, the brokerage sends the orders to stock exchange markets for execution. Figure 1 describes a general process for order submission. Bacidore et al. [1] reported that on average, the time from a market order’s exposure to its execution was approximately 22.5 seconds, while the exposure-to-execution time for a limit order is much longer. Since the exposure-to-execution time for the market order is measurable, we now develop a preference measure of submission time for market orders.

Let \( t_0 \) be the average time between an order received and the order executed; \( T_0 \) be an upper bound of the time that includes order submission and execution times (stage 1 and stage 2 in Figure 1); \( t_e \), \( t_n \) be the average duration from initiating to submitting an order at e- or non-e-brokerages. We thus define the preference measures of submission time at e- and non-e-brokerages, \( u_{2e} \) and \( u_{2n} \), as below:

\[
 u_{2e} = \frac{T_0 - (t_0 + t_e)}{T_0} \quad \text{and} \quad u_{2n} = \frac{T_0 - (t_0 + t_n)}{T_0} \quad 0 < t_e, t_n < T_0 \quad t_0
\]
 Investors Evaluate the Pros and Cons Of Brokerages

Place an Order via E-Brokerages

Order Execution at MM or EES

Place an Order via Non-E-Brokerages

Stage 1

Stage 2

Figure 1 Order Submission and Execution Process

Note:
1. MM means money market and EES means electronic exchange system.
2. Stage 1 represents the time between initiation and completion of a trade order. Stage 2 represents the time between reception and execution of a trade order.
3. Orders placed by investors can be either market orders or limited orders.

On average, $t_e < t_n$ (placing an online order at an e-brokerage is faster than contacting a broker), this means $u_{2e} > u_{2n}$. In other words, if submission time is the only criterion in choosing a brokerage, e-brokerages are preferred.

3.3 Preference Measures of Convenience, $u_{3e}$ and $u_{3n}$

Convenience is a quite broad concept that can be considered as a combination of five attributes: (1) flexible accessing time; (2) flexible locations; (3) no limit for submission frequencies per time period (e.g., a day); (4) no special equipment required; and (5) smooth submission process. To develop measures for this qualitative factor and its attributes, we may use linear formulas similar to those in (1.1) and (1.2). Thus, let $\beta_{3ej}$ and $\beta_{3nj}$ ($j = 1, 2, ..., 5$) be the preference measures of $j$th attribute in convenience at e- and non-e-brokerages and $\alpha_{3kj}$ ($\sum_{j=1}^{5} \alpha_{3kj} = 1, k = e, n$) be the weight of $\beta_{3ej}$ and $\beta_{3nj}$. Then the preference measures of convenience at e- and non-e-brokerages are as follows:

1 For limited orders, we may use following formulas: $u_{3e} = \frac{T_o}{T_o + t_e}$ and $u_{3n} = \frac{T_o}{T_o + t_n}$.
3.4 Preference Measures of Provided Service, $u_{3e}$ and $u_{3n}$

The services provided by brokerages can be summarized into five attributes: (1) “the best price” discovery, (2) reliability in execution, (3) professional consultation and advice, (4) availability of information, and (5) quick confirmation of order execution. Among the five, “the best price” discovery seems most important to most investors. In comparison with e-brokerages, full service brokerages are able to provide greater professional investment advice, proprietary investment research, access to initial public offerings (IPOs) and capital commitment in positioning a particular stock. Leading brokers also provide some free real-time quotes, margin lending and cash management services. On the other hand, e-brokerages are able to provide a large amount of information and update it quickly. Some leading e-brokerages have now formed alliances with full brokerages to narrow the gaps in their services.

To measure the preference of provided services perceived by investors at e- and non-e-brokerages, we define $\beta_{4kj}$ ($j = 1, 2, ..., 5$, $0 < \beta_{4kj} < 1$, $k = e, n$) as the preference measure of $j$th attribute and $\alpha_{4kj}$ ($j = 1, 2, ..., 5$, $0 \leq \alpha_{4kj} \leq 1$, $\sum_{j=1}^{5} \alpha_{4kj} = 1$, $k = e, n$) be the weight (or priority) assigned to $\beta_{4kj}$ ($k = e, n$). Then the preference measures of provided service at e- and non-e-brokerages can be expressed as:

$$u_{3e} = \sum_{j=1}^{5} \alpha_{3ej} \beta_{3ej} \quad \text{and} \quad u_{3n} = \sum_{j=1}^{5} \alpha_{3nj} \beta_{3nj}$$

Clearly, if $u_{3e} > u_{3n}$, investors prefer online trading to non-e-brokerages due to the relatively high convenience factor of the former.

3.5 Preference Measure of Trading Security, $u_{5e}$ and $u_{5n}$

As we mentioned earlier, security is the major barrier preventing investors from trading online. Six attributes are important in security: (1) data
discretion, (2) personal authentication, (3) access control, (4) data integrity, (5) digital signature, and (6) non-repudiation. Similarly, we define $\beta_{5kj}$ ($j = 1, 2, ..., 6; 0 \leq \beta_{5kj} \leq 1; k = e, n$) as the measure of $j$th attribute in trading security, and $\alpha_{5kj}$ ($j = 1, 2, ..., 6, k = e, n$) as the weight of $\beta_{5kj}$ ($0 \leq \beta_{5kj} \leq 1$). The preference measures of trading security at e- and non-e-brokerages can be described as:

$$u_{5e} = \sum_{j=1}^{6} \alpha_{5ej} \beta_{5ej} \quad \text{and} \quad u_{5n} = \sum_{j=1}^{6} \alpha_{5nj} \beta_{5nj}$$

Once the preference measures of the five factors, $u_{ile}$ and $u_{in}$ ($i = 1, 2, ..., 5$), are developed, overall preferences of e- and non-e-brokerages, $u_e$ and $u_n$ in (1.1) and (1.2), can be obtained after the weights for the five factors, $w_{ile}$ and $w_{in}$ ($i = 1, 2, ..., 5$), are assigned. We will discuss the assignment of values to the factors, their attributes as well as their weights in Section 5.

In summary, based on the MAU theory, we develop single scalars to measure the overall preferences of e- or non-e-brokerages perceived by investors. The preference measures will help us in the next section to develop a stochastic order-switching model to study the long run impact of investors’ preferences of brokerages on the orders submitted to two types of brokerages.

4. A Stochastic Order-Switching Model

In the last section, we define $u_e$ and $u_n$ as overall preference measures of e- and non-e-brokerages observed by investors. Practically we believe that $u_e$ and $u_n$ vary with individuals and time. Since we are interested in studying the long-term impact of e-brokerages, we assume $u_e$ and $u_n$ are the functions of time. The assumption is acceptable because with increasing convenience and decreasing commission fees over the time, investors’ preferences for the brokerages will change gradually and dynamically. Hence we assume both $u_e$ and $u_n$ have positive limits at $\lim_{t \to \infty} u_e(t) = u_e > 0$ and $\lim_{t \to \infty} u_n > 0$. In this study, a finite-state birth and death stochastic model, called order-switching model, is developed to illustrate the sample path of the order switching process. Our analytical results verify the qualitative discussions in the early literature. In addition, to examine the impact of dynamic changes of investors’ preference, simulation can be performed on
our model by varying parameters. Before we present our model, we provide
following notation and assumptions.

**Notation:**

- \( t \): time; a parameter of the process.
- \( N \): a constant sample size observed over \( t \ (t \geq 0) \) in the process of
  investors’ order submission in the stock market, i.e. the number of state of the
  order-switching process.
- \( X(t) \): the random variable that represents the total possible number of
  orders in \( N \) submitted to non-e-brokerages at time \( t \). That is, the investors
  choose non-e-brokerages and submit \( X(t) \) orders.
  \( x \): the value of \( X(t) \) at time \( t \), \( X(t) = x \).
- \( u_n(t) \): non-e-brokerage rate. Probability that \( X(t) \) increases by 1 during a
  short time \( \Delta t \), is proportional to \( u_n(t) \Delta t \).
- \( Y(t) \): the random variable that represents the total possible number of
  orders in \( N \) submitted to e-brokerages at time \( t \). That is, the investors choose
  e-brokerages and submit \( Y(t) \) orders.
  \( y \): the value of \( Y(t) \) at time \( t \), \( Y(t) = y \) and \( x + y = N \), \( t \geq 0 \).
- \( u_e(t) \): e-brokerage rate. Probability that \( Y(t) \) increases by 1 during a
  short time, \( \Delta t \), is proportional to \( u_e(t) \Delta t \).
- \( P_x(t) \): the probability of \( \mathbb{P} \{ X(t) = x \mid X(0) = N \} \).

In the following discussion, for simplicity, \( u_n(t) \) and \( u_e(t) \) are replaced
by \( u_n \) and \( u_e \). This substitution doesn’t change our analytical results of the
long-run trends of order submission.

**Assumptions:**

1. The order-switching process \( X(t) \), or process of choosing broker ages,
   is a birth-and-death process with a finite state space of \( N \). \( u_n \), the preference
   for the non-e-brokerage, represents the birth rate of \( X(t) \) (or the death rate of
   \( Y(t) \)). \( u_e \), the preference for the e-brokerage, represents as the death rate of
   \( X(t) \) (or the birth rate of \( Y(t) \)). The order switching depends on (i) rates of
   \( u_n \) and \( u_e \), and (ii) the historical information of the number of orders
   submitted to non-e-brokerages, \( x \ (x \leq N) \).
2. The transition probability in the order-switch process can be charac-
terized by $\Delta t$, $u_n$, $u_e$ and $x$, given that $t \geq 0$, $X(t) + Y(t) = x + y = N$, the following conditions hold:

$$P\{X(t + \Delta t) = x+1 | X(t) = x\} = u_n x \Delta t + o(\Delta t) \quad (3.1)$$

$$P\{X(t + \Delta t) = x-1 | X(t) = x\} = u_e (N-x) \Delta t + o(\Delta t) \quad (3.2)$$

$$P\{|X(t + \Delta t) - X(t)| > 1\} = o(\Delta t) \quad (3.3)$$

By adding (3.1), (3.2) and (3.3), we have

$$P\{X(t + \Delta t) = x | X(t) = x\} = 1 - [u_n x \Delta t + u_e (N-x) \Delta t] + o(\Delta t) \quad (3.4)$$

3. The initial conditions are: $X(0) = nN$ and $Y(0) = eN$ with $nN + eN = N$.

Note that $X$ varies with the investors’ choice of brokerages. The probability in (3.1) describes the transition process in which if one more investor chooses the non-e-brokerage and submits an order in time interval $(t, t + \Delta t)$, $x$ increases by 1. This implies that $y$ decreases by 1 ($x + y = N$). Similarly, the probability in (3.2) describes the process for an investor to submit an order to e-brokerage in time interval $(t, t + \Delta t)$ so that $x$ decreases by 1 and $y$ increases by 1 ($x + y = N$). The Probability of (3.3) provides the possibility of more than one orders switched between e- and non-e-brokerages in $(t, t + \Delta t)$. The probability of (3.4) indicates no change of $x$ in $(t, t + \Delta t)$. Figure 2 describes the sample path of the order-switch process. Since $N$ is finite, we only need to model the sample path of $X(t)$ for $t \geq 0$. That is, the total number of orders in $N$ submitted to non-e-brokerages.

For Assumption 2, we note when $x = N$ (or $x = 0$), the switch process will be interrupted, i.e. as long as non-e-brokerage (or e-brokerage) have absorbed all the $N$ orders at time $t$, then the e-brokerage (or non-e-brokerage) is incapable of absorbing orders any more in future $t + \Delta t$. In case this situation occurs and let the switch process continue forward, we use $x + 1$, $y+1$ in the right side of transition probability equations (3.1), (3.2) and (3.4), instead of $x$, $y$. Thus the transition probability, for example, $P\{X(t + \Delta t) = x+1 | X(t) = x\} = u_n (x + 1) \Delta t + o(\Delta t)$, $P\{X(t + \Delta t) = x-1 | X(t) = x\} = u_e (N-x + 1) \Delta t + o(\Delta t)$

Orders via Brokers ($x$)

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In practical problem, this situation can be understand that the trading market exists a market keeper who submit orders through non-e-brokerage and e-brokerage in order to maintain the market operation and to absorb the investors’ orders submitted through both non-e-brokerage and e-brokerage. The market keeper’s order submission is independent of the priority of non-e-brokerage and e-brokerage. Its orders are not considered as observations. There is one and only one order taken from e- and non-e-brokerage respectively is always considered in the models, but not included in the N observed orders.

Furthermore, since $P\{X(0) = N_n\} = 1$, we replace $P \{X(t) = x | X(0) = N_n\}$ by $P_n(t)$ with $P_n(t) = P \{X(t) = x\}$. The transition functions of the order-switching process are as follows:

$$
\begin{align*}
P_n(t + \Delta t) &= P \{X(t+\Delta t) = x\} \\
&= P \{X(t+\Delta t) = x | X(t) = x-1\} P \{X(t) = x-1\} \\
&\quad + P \{X(t+\Delta t) = x | X(t) = x+1\} P \{X(t) = x+1\} \\
&\quad + P \{X(t+\Delta t) = x | X(t) < x-1 \text{ or } X(t) > x+1\} P \{X(t) < x-1 \text{ or } X(t) > x+1\} \\
&= u_n(x-1+1)\Delta t P_{x-1}(t) + u_e(N-(x+1)+1)\Delta t P_{x+1}(t) + \left[1 - [u_n(x + 1) + u_e(N-x + 1)]\Delta t\right] P_x(t) + o(\Delta t)
\end{align*}
$$

Figure 2 Order Switching Process
When $\Delta t \to 0$, a system of differential difference equations for transition probability functions can be expressed as:

$$
\frac{dP_x(t)}{dt} = u_n P_{x-1}(t) + u_e (N-x)P_{x+1}(t) - [u_n(x+1) + u_e(N-x+1)]P_x(t)
$$

where $1 < x \leq N$.

(4.1)

$$
\frac{dP_1(t)}{dt} = u_e P_1(t) - u_n P_0(t)
$$

(4.2)

$$
\frac{dP_N(t)}{dt} = u_n N P_{N-1}(t) - u_e P_N(t)
$$

(4.3)

(4.1), (4.2) and (4.3) are a system of differential-difference equations. In the next two sections, we will derive asymptotic or limiting results for two cases: (i) investors choose brokerages depending only on $u_e$ and $u_n$; and (ii) investors make choices based on the historical information about total orders at brokerages and $u_e$, $u_n$. The asymptotic analyses will help us to identify long-term trend of orders in the stock market.

4.1 Choice of Brokerages Depending Only On $u_e$ and $u_n$

The differential-difference equations in (4.1), (4.2) and (4.3) assume order numbers are based not only on the preference rates, $u_e$ and $u_n$, but also on the history of orders submitted to non-e-brokerages. In this section, we assume investors choose their brokerages based only on $u_e$ and $u_n$. Then, equations (4.1), (4.2) and (4.3) can be simplified as:

$$
\frac{dP_x(t)}{dt} = u_n P_{x-1}(t) + u_e P_{x+1}(t) - (u_n + u_e)P_x(t)
$$

where $1 \leq x \leq N-1$.

(5.1)

$$
\frac{dP_1(t)}{dt} = u_e P_1(t) - u_n P_0(t)
$$

(5.2)

$$
\frac{dP_N(t)}{dt} = u_n N P_{N-1}(t) - u_e P_N(t)
$$

(5.3)

When $t \to \infty$, let $\pi_x = \lim_{t \to \infty} P_x(t)$. Then, $\{X(t); t \geq 0\}$ has a limiting distribution of $\{\pi_x\}$. For finite state of $X(t)$, equations (5.1) to (5.3), at $t \to \infty$, can be expressed as below:

$$
(u_n + u_e)\pi_x = u_n \pi_{x-1} + u_e \pi_{x+1}
$$

where $1 \leq x \leq N-1$.

(6.1)
Note that \( \{ x \pi \} \) is the probability distribution of the number of orders submitted to non-e-brokerages in long-term future, i.e. the distribution of \( X(t) \) when \( t \to \infty \), \( 0 \leq x \leq N \). In the later part of the paper, we will obtain the expression of \( \pi_x \) and relevant \( E_N(X) \), the expected value of \( X(t) \) when \( t \to \infty \). Based on the equations (6.1) to (6.3), we have following two Lemmas.

**Lemma 1:** The limiting probability, \( \pi_x \), can be expressed as:

\[
\pi_x = \frac{u_n}{u_e} \pi_{x-1} \quad 0 \leq x \leq N
\]

**Proof:** Using induction and equations (6.1), (6.2) and (6.3), it is easy to show the above equation is true. \( \square \)

**Lemma 2:** Let \( r = \frac{u_n}{u_e} \). For a given \( N \), the expected value of \( X(t) \), when \( t \to \infty \), \( E_N(X) \), has the following expression:

\[
E_N(X) = \frac{r}{(1-r)[1-r^{N+1}]} \left[ N r^{-N+1} - (N+1)r^{-N} + 1 \right]
\]

**Proof:** See Appendix A.

Now we will calculate asymptotic results when \( t \to \infty \) and for enough large market size of \( N \). Propositions 1 to 3 describe the long run trend of the orders submitted to e- and non-e-brokerages for given preference measures, \( u_e \) and \( u_n \).

**Proposition 1:** If \( r = \frac{u_n}{u_e} < 1 \) (\( u_e > u_n \)), then, when \( N \to \infty \), \( \lim_{N \to \infty} \frac{E_N(X)}{E_N(Y)} \to 0 \) or \( \lim_{N \to \infty} \frac{E_N(Y)}{N} = 1 \). This means online orders will gradually dominate the stock market and orders to non-e-brokerages will finally diminish with the time. That is, in a long run, all the investors in the market will eventually choose e-brokerage.

**Proof:** See Appendix B.

**Proposition 2:** If \( r > 1 \) (\( u_n > u_e \)), then \( \lim_{N \to \infty} \frac{E_N(X)}{N} = 1 \). The orders submitted to non-e-brokerages will dominate the stock market and online orders will
finally diminish with the time. That is, in a long run, all the investors in the market will eventually choose non-e-brokerage.

**Proof:** See Appendix B.

**Proposition 3:** If $r = 1$ ($u_n = u_e$), then, $E_N(X) = \sum_{x=0}^{N} x \pi_x = \frac{N}{2}$. That is, non-e-brokerages and e-brokerages will equally share the total numbers of orders in the stock market with the time.

**Proof:** See Appendix B.

Although the steady-state results obtained in Propositions 1, 2 and 3 are not surprising and are consistent with the qualitative assessment by early literature, our theoretical results analytically show that as long as the overall preference measure of e-brokerages is superior to that of non-e-brokerages, the orders submitted to non-e-brokerages will diminish in the long run. In Section 5, we will further examine how the factors and attributes interact each other and affect investors’ choice of brokerages and the long-term trend of orders in the market as well.

### 4.2 Choice of Brokerages with Both Historical Information and Preference of $u_e, u_n$

We extend the order-switching model to the case in which investors not only have their personal preference measures of e- and non-e-brokerages but are also informed by the number of orders submitted to brokerages. It means that the models comprise both their preference to the two types of broker-ages and their experience of brokerage choice in order submission.

Since $\pi_x = \lim_{t \to \infty} P_x(t)$ and following equations (4.1), (4.2) and (4.3), we obtain:

\[ \mu_n x \pi_{x-1} + u_e (N-x) \pi_{x+1} = [u_n (x+1) + u_e (N-x+1)] \pi_x \quad 0 \leq x \leq N-1 \quad (7.1) \]

\[ \pi_N = \frac{u_n}{u_e} \pi_0 \quad (7.2) \]

\[ \pi_N = \frac{u_n}{u_e} \pi_{N-1} \quad (7.3) \]

Notice that (7.1), (7.2) and (7.3) include variable of $x$ that represents the number of the orders in $N$ submitted to non-e-brokerages. We now propose Lemmas 3, 4 and 5 to derive theoretical results.
Lemma 3: With the information of history about orders submitted to non-e-brokerages, the limiting probability distribution satisfies following conditions:

\[ \pi_{x+1} = \frac{u_x (x+1)}{u_e (N-x)} \pi_x \quad \text{or} \quad \pi_x = r^x \frac{1}{\binom{x}{N}} \pi_0 \quad \text{with} \quad \binom{x}{N} = \frac{N!}{x!(N-x)!} \quad 0 \leq x \leq N-1 \]

Proof: See Appendix A.

Lemma 4: For given \( N \), let \( S_N = \sum_{x=0}^{N} r^x \frac{1}{\binom{x}{N}} \). The expected value of \( X(t) \), \( E_N(X) \), when \( t \to \infty \) satisfies following condition:

\[ E_N(X) = \frac{N+1}{r} \frac{S_{N+1}}{S_N} - \frac{N+1}{r} \frac{1}{S_N} \cdot 1 \]

Proof: See Appendix A.

Lemma 5: Given that \( S_N = \sum_{x=0}^{N} r^x \frac{1}{\binom{x}{N}} \), \( S_N \) can be expressed by following values for \( r = \frac{u_x}{u_e} \):

\[
\begin{cases}
1 & r < 1 \\
2 & r = 1 \\
\infty & r > 1
\end{cases}
\]

\[
\begin{cases}
1 & r \leq 1 \\
r & r > 1
\end{cases}
\]

Proof: See Appendix A.

We now present asymptotic results to describe the long run trend of order submissions by investors when they make choices of brokerages based on (i) their preference measures; and (ii) historical information about the orders submitted to non-e-brokerages.

Proposition 4: If \( r < 1 \), then \( \lim_{N \to \infty} E_N(X) = 0 \). That is, online orders will dominate the stock market and orders to the non-e-brokerages will finally
diminish with time. In the long-term brokerage choice process, the investors will finally choose e-brokerage to submit their orders, and further, the historical information about orders or investors’ experience in brokerage choice accelerates the domination of online orders in the market.

**Proof:** See Appendix B.

**Proposition 5:** If \( r > 1 (u_R > u_e) \), then \( \lim_{N \to \infty} \frac{E_N(X)}{N} = 1 \). That is, the orders submitted to non-e-brokerages will dominate the stock market and online orders will finally diminish with time. In the brokerage choice process, the investors will finally choose non-e-brokerage to submit their orders.

**Proof:** See Appendix B.

**Proposition 6:** If \( r = 1 (u_R = u_e) \), then \( \lim_{N \to \infty} \frac{E_N(X)}{N} = \frac{1}{2} \). That is, the orders submitted to non-e-brokerages and e-brokerages will equally share the total number of orders in the stock market. In the process of investors’ brokerage choice, or the process of order competition between two types of brokerages, each type of brokerage will capture 50% of the total orders in the market.

**Proof:** See Appendix B.

We should point out although Propositions 4, 5 and 6 are very similar to Propositions 1, 2 and 3, our analytical results show that with the historical information, the order-switching process accelerates. In other words, the expected number of orders will approach the asymptotic limits faster than that when historical information about the orders submitted to brokerages is unavailable. This results could be a theoretical explanation for earlier survey on investors’ behaviors in online trading that, comparison to the inexperienced, the investors experienced and successful in stock market more quickly chose the e-brokerage, switched from non-e-brokerages. Section 5 will present tables and figures to illustrate the differences between the two cases.

In conclusion of this section, we provide analytical results for the order-switching model considering both investors’ preferences of measures and historical information. We show that, in the long run and for a large enough market size (i.e. when \( t \to \infty \) and \( N \to \infty \)), there are three possibilities: (i) online orders dominate the stock market; (ii) orders to non-e-brokerages dominate the stock market; and (iii) both online and non-online orders share
the stock market. In the next section, we discuss the empirical results of extensive computational experiments.

5. Design of Computational Experiments

In this section, we first discuss the development of the measures of factors, attributes and weights in (1.1), (1.2), (2.1) and (2.2) and then discuss computational results. The two linear formulas, (1.1) and (1.2), developed in Section 3 measure overall investors’ preferences of e- or non-e-brokerages and the by incorporating five factors: commission, submission time, convenience, provided service and trading security. For the five factors, each of convenience provided service and trading security has five to six attributes. In our computational experiments, we need to generate the measures for all the factors, their attributes, and their weights. By searching earlier literature and using our own judgment, we came up a set of values for the factors, attributes and weights that are presented in Table 1.

While all the attributes are assigned values from 0 to 1, the sum of the weights of attributes and factors are equal to 1.0. To describe different trading activities (say, market and limit orders), we used three different sets of weights to represent investors’ preferences of the factors in market, limit and integrated orders (The integrated order combines the market and the limit orders together. The measures of the factors in the integrated order are the averages of preference measures of the market and the limit orders).

Following the three sets of weights, we obtained three different values: $r = 1.07$, $r = 0.93$ and $r = 0.99$. When $r = 1.07$, it means overall investors prefer non-e-brokerages. When $r = 0.93$, it means investors prefer e-brokerages and when $r = 0.99$, the investors slightly prefer e-brokerages. In other words, when $r = 1.07$, it represents investors’ preference of full service brokerages in market orders. When $r = 0.93$, it represents investors’ preference of e-brokerages in limit orders. And $r = 0.99$ represents integrated orders.

Note that although the development of the measures and weights for factors, attributes and weights are subjective, changing the values of factors, attributes and weights would not change the theoretical results. The computational results are only used to illustrate the asymptotic results of investors’ choices of brokerages based on their preferences and historical information about the orders submitted to brokerages.
### Table 1 Design of Parameters

<table>
<thead>
<tr>
<th>Factor</th>
<th>E-Brokerages</th>
<th>Non-E-Brokerages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commission</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>Submission Time</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Convenience</th>
<th>E-Brokerages</th>
<th>Non-E-Brokerages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexible Access</td>
<td>0.9</td>
<td>0.5</td>
</tr>
<tr>
<td>Flex Location</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>Flexible Frequency</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>No Special Equipment</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Smooth Submission Process</td>
<td>0.7</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Provided Service</th>
<th>E-Brokerages</th>
<th>Non-E-Brokerages</th>
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</thead>
<tbody>
<tr>
<td>Best Price</td>
<td>0.3</td>
<td>0.8</td>
</tr>
<tr>
<td>Discovery</td>
<td>0.3</td>
<td>0.9</td>
</tr>
<tr>
<td>Reliability in Execution</td>
<td>0.1</td>
<td>0.8</td>
</tr>
<tr>
<td>Consultation &amp; Advice</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>Information Availability</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>Quick Confirmation</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Security</th>
<th>E-Brokerages</th>
<th>Non-E-Brokerages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Discretion</td>
<td>0.2</td>
<td>0.9</td>
</tr>
<tr>
<td>Personal Authentication</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>Access Control</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>Data Integrity</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>Digital Signature</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>Non-repudiation</td>
<td>0.7</td>
<td>0.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Weight Set</th>
<th>0.3</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commission</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Submission Time</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Convenience</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Provided Service</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Security</td>
<td>0.65</td>
<td>0.61</td>
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<td></td>
</tr>
<tr>
<td>5.1 Computational Results of Order-Switching Models</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In our computational experiments, we first examine the asymptotic results of expected orders to non-e-brokerages for various $N$. Figures 3 and 4 present the results of $\frac{E(X)}{N}$ by varying $N$ from 10 to 500. Propositions 1 and 2 state that when $N \to \infty$ and $t \to \infty$, $\frac{E(X)}{N}$ will approach its limit at 1 or 0. If investors prefer non-e-brokerages ($r > 1$), $\frac{E(X)}{N} \to 1$ and orders to
e-brokerages will diminish. If investors prefer e-brokerages \((r < 1)\), \(\frac{E(X)}{N} \rightarrow 0\) and orders to non-e-brokerage will diminish. In Figure 3, we notice that when \(r = 1.07, 0.99\) and \(0.93\) and \(N\) is small, \(\frac{E(X)}{N}\) are not close to 1 or 0.

Only when \(N = 500\), \(\frac{E(X)}{N}\) is close to 1 or 0. Figure 4 shows when investors have both preference and historical information about orders to non-e-brokerages and \(r = 0.93\) or \(1.07\), \(\frac{E(X)}{N}\) approaches 0 or 1 when \(N\) is close to 70. This indicates that the historical information about order status or investors’ experiences in brokerage choice accelerates the domination of the orders to the non-e-brokerage or e-brokerage. It’s also interesting to note that when \(r = 0.99\), the converging process, \(\frac{E(X)}{N} \rightarrow 0\), is slower than that when \(r = 0.93\) or \(1.07\). This indicates that when the preference measures of e- and non-e-brokerages are very close, investors tend to be slow in switching brokerages.

![Figure 3 Order Distribution-Submission Depending on \(u_a\) and \(u_e\)](image-url)
Figures 5 and 6 examine the impact of the asymptotic results of the expected orders to non-e-brokerages with respect to various \( r \). Figure 4 presents the expected orders when investors’ choices are only based on their preferences, and Figure 5 describes expected orders when investors’ choices are based on both preferences and historical information about orders. Without historical information, the expected orders approach their limits slower than with historical information.

In summary, our computational results show there exist differences in the asymptotic results of the expected orders with and without historical information about orders submitted to the brokerages. With the information, the expected orders approach their limits much faster than without information. Intuitively, this is reasonable because additional information definitely assists investors to make quick decisions.
5.2 Sensitivity Analysis of Commissions and Security in the Order-Switching Model

While commission is the key factor that attracts investors to trade online, security is the factor that adversely affects online trading. Therefore, we focused our sensitivity analyses on these two factors. As discussed in Section 3, trading security includes six attributes. For simplicity, we selected data discretion and personal authentication as two major attributes and performed sensitivity analysis on these two attributes. In Table 1, both attributes have the same preference measure of 0.2 and the same weight of 0.3 in the trading security at e-brokerages. The results of the sensitivity analyses are presented in Table 2.

As noted above, we used three different sets of weights for the five factors and obtained \( r = 1.07, 0.93, \) and 0.99. In the case of \( r = 1.07 \) (investors prefer non-e-brokerages to e-brokerages), we gradually increase the preference measures of either data discretion or personal authentication at e-brokerages from 0.2 to 1 (see column 1 in Table 2) to study the impact of the improvement of trading security at e-brokerages. The corresponding preference measures of trading security and \( r \) values are presented in columns 2 and 3. When the attributes of data discretion or personal authentication increase to 0.7, the preference measure of security at e-brokerage is equal to 0.52 and \( r \) is equal to 1.0, which indicates orders to e- and non-e-brokerages will share the stock market. When the preference measure of data discretion or personal authentication at e-brokerages increases to 1.0, \( r = 0.961 \), which indicates that e-brokerage is now superior to non-e-brokerage. On the other hand, when \( N = 100, E_{\text{opt}}(Y) \), the expected orders to e-brokerages, increase
rapidly with the increase in values of security attributes, especially in the case of investors with historical information (see the lower parts of columns 2 and 3 in Table 2). Note that when data discretion or personal authentication increase to 1.0, \( E_{100}(Y) \) is equal to 98.15, which indicates that orders submitted to e-brokerages almost dominate the whole stock market even though \( N = 100 \) is not a large sample size. (Theoretically, for \( r < 1 \), only when \( N \to \infty \) and \( t \to \infty \), \( E(Y) \to 1 \)). In summary, our sensitivity results show that investors’ preferences and expected orders to brokerages are very sensitive to the improvement of the services provided by e-brokerages.

Table 2 Sensitivity Analysis on Security and Commissions

<table>
<thead>
<tr>
<th>Initial Status: ( r = 1.07 )</th>
<th>Initial Status: ( r = 0.93 )</th>
<th>Initial Status: ( r = 0.99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increase of One Attribute: Either Data Discretion or Personal Authentication</td>
<td>Preference Measure of Security at E-Brokerages</td>
<td>Decrease of Commissions at E-Brokerages</td>
</tr>
<tr>
<td>( E_{100}(Y) ) ( = \frac{a}{b} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td>Preference Measure of Commissions at Non-E-Brokerages</td>
</tr>
<tr>
<td>( E_{100}(X) ) ( = \frac{c}{d} )</td>
<td>Preference Measure of Commissions at Non-E-Brokerages</td>
<td>Value of ( r = \frac{a}{b} )</td>
</tr>
<tr>
<td>( E_{100}(X) ) ( = \frac{e}{f} )</td>
<td>Preference Measure of Commissions at Non-E-Brokerages</td>
<td>Value of ( r = \frac{a}{b} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Initial Status: ( r = 1.07 )</th>
<th>Initial Status: ( r = 0.93 )</th>
<th>Initial Status: ( r = 0.99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increase of One Attribute: Either Data Discretion or Personal Authentication</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
</tr>
<tr>
<td>( E_{100}(Y) ) ( = \frac{a}{b} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
</tr>
<tr>
<td>( E_{100}(X) ) ( = \frac{c}{d} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
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<tr>
<td>( E_{100}(X) ) ( = \frac{e}{f} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
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<table>
<thead>
<tr>
<th>Initial Status: ( r = 1.07 )</th>
<th>Initial Status: ( r = 0.93 )</th>
<th>Initial Status: ( r = 0.99 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increase of One Attribute: Either Data Discretion or Personal Authentication</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_{100}(Y) ) ( = \frac{a}{b} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
</tr>
<tr>
<td>( E_{100}(X) ) ( = \frac{c}{d} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
</tr>
<tr>
<td>( E_{100}(X) ) ( = \frac{e}{f} )</td>
<td>Decrease of Commissions at E-Brokerages</td>
<td></td>
</tr>
</tbody>
</table>

| Increase of One Attribute: Either Data Discretion or Personal Authentication |
| \( E_{100}(Y) \) \( = \frac{a}{b} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{c}{d} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{e}{f} \) | Decrease of Commissions at E-Brokerages |

| Increase of One Attribute: Either Data Discretion or Personal Authentication |
| \( E_{100}(Y) \) \( = \frac{a}{b} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{c}{d} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{e}{f} \) | Decrease of Commissions at E-Brokerages |

| Increase of One Attribute: Either Data Discretion or Personal Authentication |
| \( E_{100}(Y) \) \( = \frac{a}{b} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{c}{d} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{e}{f} \) | Decrease of Commissions at E-Brokerages |

| Increase of One Attribute: Either Data Discretion or Personal Authentication |
| \( E_{100}(Y) \) \( = \frac{a}{b} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{c}{d} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{e}{f} \) | Decrease of Commissions at E-Brokerages |

| Increase of One Attribute: Either Data Discretion or Personal Authentication |
| \( E_{100}(Y) \) \( = \frac{a}{b} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{c}{d} \) | Decrease of Commissions at E-Brokerages |
| \( E_{100}(X) \) \( = \frac{e}{f} \) | Decrease of Commissions at E-Brokerages |
In the case of $r = 0.93$ or $0.99$, investors prefer e-brokerages to non-e-brokerages. To study whether non-e-brokerages can improve their service by reducing commissions, we performed sensitivity analysis on commissions. We gradually reduced the commissions at non-e-brokerages from $75$ to $35.95$ for initial $r = 0.93$ and from $75$ to $54.4$ for initial $r = 0.99$. In Table 2, we observe that for the initial $r = 0.93$ and $0.99$, $r$ becomes $1$ when the commissions at non-e-brokerages reduce to $51.87$ and $72.13$ respectively. We also note that with historical information on orders, $E_{10}(X)$ increases to $99.23$ when the commissions reduce to $35.95$ and $54.4$ respectively. Our sensitivity results suggest that even though investors prefer online trading, if non-e-brokerages reduce their commissions to certain prices, investors will come back to non-e-brokerages.

6. Conclusion and Extension

In this paper, we have studied the long-run impact of investors’ choice of brokerages on the stock market. In recent years there has been an explosion in online trading that is likely to continue in the next decade. There are many advantages and disadvantages to online trading. To study how and why investors switch brokerages, we identified five important factors that affect the investors’ choice of brokerages. Since some factors are qualitative, we developed linear formulas to convert multi-factors and multi-attribute measures into scalars to represent investors’ overall preferences of brokerages. We focused on two types of brokerages: e-brokerage and non-e-brokerage.

Based on the investors’ preference measures of brokerages, a stochastic process called order-switching model was then developed to study the long-term trend of investors’ orders in the stock market. Both analytical and empirical results were derived. Our results show when basing their choice only on preference measures, investors are slower in switching brokerages than when basing their choice on both preference measures and historical information on orders submitted to each type of brokerage. In the long run, if the preference measure of one type of brokerage is superior to the other type, the orders submitted to the superior type of brokerage will dominate the stock market. Our sensitivity analyses indicate that as soon as the inferior brokerages improve their major weak factor significantly, the orders submitted to them will increase quickly.

This research can be extended in many directions. One direction is to collect empirical data to derive more practical measures of factors and
attributes and use them to test our order-switching model. Another direction is to extend our current model to incorporate more interesting issues such as dynamic preference measures and competition strategies implemented by e- and non-e-brokerages. Finally, a more comprehensive model can be developed to examine the financial impact of the order-switching process on the stock market.

References


Appendix A: Proofs of Lemma 2 to 5

Proof of Lemma 2:

Since \( r = \frac{\mu_x}{\mu_e} \) and from Lemma 1, we know \( \pi_x = r^x \pi_0 \). Since \( \{ \pi_x \} \) is the limiting probability distribution, \( \sum_{x=0}^{N} \pi_x = \sum_{x=0}^{N} r^x \pi_0 = 1 \). Then, we obtain \( \pi_0 = \frac{1-r}{1-r^{N+1}} \) and \( \pi_x = r^x \frac{1-r}{1-r^{N+1}} \). Hence, \( E(X) = \sum_{x=0}^{N} x \pi_x = \frac{1-r}{1-r^{N+1}} \sum_{x=0}^{N} x r^x = \frac{1-r}{1-r^{N+1}} \left[ N r^{N+1} - (N+1) r^N + 1 \right] \). \( \Box \)

Proof of Lemma 3:

By using induction, for \( x = 0 \), following equation (7.2), we know the lemma is true. If it is true for \( x = k \), i.e. \( \pi_k = \frac{u_e(N-K)}{u_e(K+1)} \pi_{k+1} \) or \( \pi_k = \frac{u_e(N-K)}{u_e(K+1)} \pi_{k+1} \), then for \( x = k+1 \), following equation (7.1), we have

\[
\begin{align*}
\pi_{k+1} &= \left\{ u_e[(k+1)+1] + u_e[N-(k+1)+1] \pi_{k+1} - u_e(k+1) \pi_k \right\} \pi_{k+1} \\
&= \left\{ u_e[(k+1)+1] + u_e[N-(k+1)+1] \pi_{k+1} - u_e(k+1) \frac{u_e(N-K)}{u_e(K+1)} \pi_{k+1} \right\} \pi_{k+1} \\
&= u_e[(k+1)+1] \pi_{k+1} \\
\pi_{(k+1)+1} &= \frac{u_e((K+1)+1)}{u_e(N-(K+1))} \pi_{K+1} 
\end{align*}
\]
Thus \( \pi_{x+1} = \frac{u_x(x+1)}{u_x(N-x)} \pi_x \) for \( 0 \leq x \leq N - 1 \). And therefore, we obtain

\[
\pi_x = r^x \frac{1}{N} \pi_0. \quad \square
\]

**Proof Lemma 4:**

Since \( \pi_x = r^x \frac{1}{N} \pi_0 \) (see Lemma 3) and \( \sum_{x=0}^{N} \pi_x = 1 \), then

\[
\sum_{x=0}^{N} \pi_x = \sum_{x=0}^{N} r^x \frac{1}{N} \pi_0 = 1. \text{ Therefore, } \pi_0 = \frac{1}{N} \text{ and } E_{\pi}(X) = \sum_{x=0}^{N} x \pi_x =
\]

\[
\sum_{x=0}^{N} x r^x \frac{1}{N} \pi_0 = \frac{1}{N^r} \sum_{x=0}^{N} x r^x = \frac{1}{N^r} \sum_{x=0}^{N} x \frac{1}{N} = \frac{1}{N} \left[ \frac{(N+1)^2}{2} \right] = \frac{1}{N} \left[ \frac{(N+1)^2}{2} \right].
\]

\[
= \frac{N+1}{N} \left( \frac{S_N}{N} \right) - \left( \frac{N+1}{N} \right) \frac{1}{S_N} - 1 \quad \square
\]

**Proof of Lemma 5:**

Since \( S_N = \sum_{x=0}^{N} x r^x \frac{1}{N} = 1 + r \frac{1}{N} + r^2 \frac{1}{N} + \cdots + r^{N-1} \frac{1}{N} \), we know

\[
\frac{1}{N} \frac{(N+1)^2}{2} < \frac{2}{N(N-1)} \quad \text{for } 2 \leq x \leq \frac{N}{2}.
\]

Furthermore, using \( \frac{x}{N} = \frac{N-x}{N} \), we obtain \( \frac{1}{N} \frac{(N-x)}{N} \leq \frac{2}{N(N-1)} \), for \( 2 \leq x \leq N-2 \).

This means that for any \( r \leq 1 \), \( \sum_{x=2}^{N-2} \frac{1}{N} x r^x \to 0 \) with \( N \to \infty \). Thus, when \( N \to \infty \), given that \( S_N = \sum_{x=0}^{N} x r^x \frac{1}{N} = 1 + r \frac{1}{N} + r^2 \frac{1}{N} + \cdots + r^{N-2} \frac{1}{N} \), we know

\[
\sum_{x=0}^{N} x r^x \frac{1}{N} = 1 + r \frac{1}{N} + \cdots + r^{N-2} \frac{1}{N} \to 0 \quad \text{as } N \to \infty.
\]
\( S_N \to 1 \) for \( r < 1 \), \( S_N \to 2 \) for \( r = 1 \) and \( \lim_{N \to \infty} \frac{S_{N+1}}{S_N} = 1 \) for \( r \leq 1 \). Thus (8.1) to (8.2) and (8.4) are true. It is apparent that (8.3) is true.

To show that \( \lim_{N \to \infty} \frac{S_{N+1}}{S_N} = r \) \( (r > 1) \), consider

\[
\frac{S_N}{r^N} = \sum_{x=0}^{N-r} \frac{1}{x} = \sum_{x=0}^{N-r} \frac{1}{N^x} \sum_{x=0}^{N-r} \frac{1}{N^x}.
\]

Note that the expression of \( \sum_{x=0}^{N-r} \frac{1}{x} \) is the same as \( S_N \) except it has \( \frac{1}{r} \) instead of \( r \) in \( S_N \). Since \( \frac{1}{r} < 1 \), by (8.1), we know \( \frac{S_N}{r^N} = \sum_{x=0}^{N-r} \frac{1}{x} \to 1 \).

So \( \lim_{N \to \infty} \frac{S_{N+1}}{S_N} = r \) for \( r > 1 \), that is, (8.5) is true.

Appendix B: Proofs of Propositions 1 to 6

Proof of Proposition 1:

For \( r < 1 \), when \( N \to \infty \), it is clear from Lemma 2 that \( \lim_{N \to \infty} \frac{E_N(X)}{N} = \frac{E_N(Y)}{N} = \frac{r}{1-r} \).

This means the expected number of submitted orders to non-e-brokerages is a small value. Let \( E_N(Y) \) be the expected order number to e-brokerages. Since \( E_N(Y) = N - E_N(X) \), then \( \lim_{N \to \infty} \frac{E_N(X)}{E_N(Y)} \to 0 \) or \( \lim_{N \to \infty} \frac{E_N(Y)}{N} = E_N(X) / N = 1 \). In contrast to \( E_N(Y) \), \( E_N(X) \) is rather small when \( N \) is large enough. Hence we say that online orders will dominate the stock market.

Proof of Proposition 2:

Since \( E_N(X) = \frac{r}{(1-r)(1-r^N)} \left[ 1 - (N+1)r^{-N+1} + 1 \right] \), with a little algebra, we may obtain:

\[
E_N(X) = \frac{r}{(1-r)(1-r^N)} \left[ N \frac{r^{-1}}{r} + r^{-N} \right] = \frac{1}{(1-r)(1-r^N)} \left[ N \frac{r^{-1}}{r} + r^{-N} \right].
\]
Note \( E_N(X) \) can also be expressed as \( N - \frac{1}{1-r} + \varepsilon_N \), where \( \varepsilon_N = o\left(\frac{1}{N}\right) \) for \( r > 1 \). Thus \( \lim_{N \to \infty} \frac{E_N(X)}{N} = 1 \). This means that when \( N \) and \( n \) are large enough, the order numbers submitted to non-e-brokerages will dominate the market. \( \Box \)

Proof of Proposition 3:
Following Lemma 1, \( \pi_x = \frac{u_x}{u_e} \pi_{x-1} \), for \( 0 \leq x \leq N \). Since \( r = 1 \), then \( \pi_1 = \pi_0 \) for \( 0 \leq x \leq N \).

Since \( \pi_x \) is limiting probability, \( \sum_{x=0}^{N} \pi_x = 1 \). Thus, \( \pi_x = \pi_0 = \frac{1}{N+1} \) for \( 0 \leq x \leq N \). Therefore, \( E_N(X) = \sum_{x=0}^{N} x \pi_x = \frac{N}{2} \). \( \Box \)

Proof of Proposition 4:
Since \( \pi_x = r^x \frac{1}{x} \pi_0 \) (see Lemma 3) and \( \pi_0 = \frac{1}{S_N} \), for a given \( N \),

\[
E_N(X) = \frac{N}{x=0} \sum_{x=0}^{N} x \pi_x = \sum_{x=0}^{N} x r^x \frac{1}{x} \pi_0 = \frac{1}{S_N} \sum_{x=0}^{N} x r^x \frac{1}{x}.
\]

Since \( \sum_{x=0}^{N} x r^x \frac{1}{x} = r \frac{1}{N} + o\left(\frac{1}{N}\right) \) and \( S_N \to 1 \) for \( r < 1 \) in (8.1) of Lemma 5,

\[
E_N(X) = r \frac{1}{N} + o\left(\frac{1}{N}\right). \quad \text{Thus} \quad \lim_{N \to \infty} E_N(X) = 0.
\]

Furthermore, we know, the expectation \( E_N(X) \) is not expectation \( E_\pi(X) \) with \( r < 1 \) in the situation: choice of brokerages depending only on \( u_x \) and \( u_e \) (in section 4.1). This means that the historical information about orders or investors’ experience in brokerage choice accelerates the domination of online orders in the market. \( \Box \)

Proof of Proposition 5:
In Lemma 4, \( E_N(X) = \sum_{x=0}^{N} x \pi_x = \frac{N + 1}{r} \cdot \frac{S_{N+1}}{S_N} - \frac{N + 1}{r} \cdot \frac{1}{S_N} - 1 \). Since

\[
\frac{1}{S_N} = o\left( \frac{1}{N} \right) \text{ for } r > 1 \text{ and following (8.5) in Lemma 5, we have}
\]

\[
\lim_{N \to \infty} \frac{E_N(X)}{N} = 1. \text{ Clearly, this means that orders to non-e-brokerages will dominate the market and online orders will finally diminish if the order submissions are large enough.} \]

**Proof of Proposition 6:**

For given \( N \), by lemma 4, we have

\[
E_N(X) = \sum_{x=0}^{N} x \pi_x = E_N(X) = \frac{N + 1}{r} \cdot \frac{S_{N+1}}{S_N} - \frac{N + 1}{r} \cdot \frac{1}{S_N} - 1
\]

With (8.2) and (8.5) in Lemma 5, we know \( S_N \to 2 \) and \( \frac{S_{N+1}}{S_N} \to 1 \). Thus we have \( \lim_{N \to \infty} \frac{E_N(X)}{N} = 1/2 \). It means that the orders submitted to non-e-brokerages and e-brokerages will equally share the market with the same volumes.