Pricing an Arithmetic Average Reset Option Using the Green Function Method

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Abstract

This study investigates the pricing of the arithmetic average reset option. The option price is formulated as the solution of the Black-Scholes equation. In addition, the valuation is derived from a series of initial value problems based on the Green function through integration. Finally, the reset option price is numerically calculated. Throughout the numerical method, we can derive reset option prices of both arithmetic average and geometric average reset options. This study also presents the numerical examples for comparison.

Keywords: Reset option, Asian option, arithmetic average, green’s function

1. Introduction

A reset option is a path-dependent derivative whose strike price can be reset based on certain criteria. According to the pre-specified conditions during the pre-set dates, the option holder can specify certain features. For example, the strike price of a reset call can be adjusted on the pre-set dates if the price of the underlying asset reaches the reset threshold. Reset options have been traded for several years. Some of the derivatives are attached to structured products. In late 1996, for example, both the Chicago Board Options Exchange (CBOE) and the New York Stock Exchange (NYSE) introduced S&P 500 index bear market reset put warrants with a 3-month reset period. This was issued by the International Finance Corporation. In June, 1997, Morgan Stanley in Hong Kong issued the resettable convertible bonds. In Taiwan, Grand Cathay Securities issued six American-style reset warrants with multiple reset prices during the reset period, which were listed on the Taiwan Stock Exchange in 1998. Recently, on February 14, 2008, Chinatrust Securities issued a reset put warrant whose strike price would be adjusted to 88% of the two-day average closing price if 114% of the two-day average closing price of EVA Airways Company fell below the initial strike price of $8.08 TWD before the go-public two days. The use of the reset option has gradually been the object of study in recent years. Hence, with the increasing usage of reset options, pricing the options with reset criteria is of decisive importance. In this study, we focus on pricing the arithmetic average reset option.

Employee options with reset features have become very popular in the last decades. After Black and Scholes (1973) and Merton (1973) introduced the valuation formula for option, the Black-Scholes model has been the core framework for pricing options. Gray and Whaley (1997, 1999) examined the pricing of the S&P 500 index of put warrant with a periodic reset

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along with the warrant’s risk characteristics under the Black-Scholes economy assumptions (Black and Scholes, 1973). Furthermore, they derived an analytic solution for the reset put option and studied the reset feature of the Geared Equity Investment offered by Macquarie Bank in Australia. Recently, reset options have been applied to the employee reset option. The main objective for implementing employee reset options is that the company wants to retain human resource and then prompt the revenue of the company. Under this system, the interests of both the employees and employer are consistent. If the corporate business achievement makes a positive growth pattern, the stock price will raise, which benefits both the staff and company. Google, for example, distributed employee reset options to inspire morale. That is, when the stock price of Google falls, the employees can exercise their option in exchange for Google’s stocks. In this way, if the employees exercise the reset options with a lower stock price, they will earn profit as the stock price rises. In addition, to achieve the higher stock price, the employees will work harder, thus Google will eventually make profit through this strategy. Another example of reset option is Cosi, the sandwich shop chain in early 2004. Cosi repriced more than 800,000 options for top executives. In defence of the repricing, Cosi stated that its goal was to motivate employee as part of a turnaround effort. Another advantage of reset options is that they can protect the investor’s downside risk. Hence, the usage of the reset option is often applied to financial hedging. For European average-rate call options with the underlying asset being the exchange rate, for instance, if the average exchange rate is higher than the exercise price at maturity, the option holder can both make a profit and offset the lost in short position of the exchange rate.

Another type of exotic options is the Asian option. Asian options are options where the payoff depends on the average price of the underlying asset during some pre-specified period of time. Asian options can mitigate the possibility of stock price manipulation, especially when the time is close to the maturity. In the petrochemical market, for example, as long as the petroleum prices are set at a specific level, the profit of the petrochemical companies will not suffer abnormal fluctuations due to volatile petroleum prices; instead, they will be stabilized. Therefore, in order to implement cost control during the oil hedging process, the average price is preferred as the exercise price. Furthermore, Asian options can reduce the volatility of the underlying asset throughout the duration. In 1995, J.P. Morgan issued a Japanese principal guaranteed fund priced in US dollars. The underlying asset of this derivative is the Nikkei 300 index. The maturity of the fund is eight quarters, if the average stock price of the eight quarters is higher than the strike price on the maturity date, the return of the fund is the average stock price minus the strike price. If reset options are based on the average of the stock prices as their underlying prices, it will be termed Asian reset options. One of the thorniest problems researchers face is that the option that covers Asian and reset features, depending on the arithmetic average closing underlying price, lacks simple closed-form solutions and is more difficult to price. Thus, other methods are often applied to approximate Asian options. Boyle (1977), Kemna and Vorst (1990) and Corwin, Boyle, and Tan (1996) used Monte Carlo simulation test but the costs are high and the processes are time-consuming. The approaches of solving partial differential equation (PDE) and lattices models are also often used in pricing Asian options.

In empirical markets, the arithmetic average trigger reset options are more frequently traded than the geometric ones. Thus, the method of effectively and accurately pricing arithmetic average reset options is reasonably important. There is, however, no explicit closed-form solution for the arithmetic average reset option. The arithmetic average of underlying stock is not log-normally distributed while the underlying stock price follows a standard log-normal process. Since there is no closed-form solution for pricing the arithmetic average reset option, numerical techniques must be used to price these options. Turnbull and Wakeman (1991) recognized the suitability of the log-normal as a first-order approximation.
They applied the Edgeworth Series in statistic theory and expanded the probability distribution of the arithmetic average. Furthermore, they approximate the value of the Asian option via using log normal distribution. Levy (1992) applied the Wilkinson approximation to price European currency options. The Monte Carlo simulation method is of the numerical procedures and it is usually used for derivatives that their payoff is dependent on the history of the underlying variables. The geometric average reset option price can be used as a control variate in the Monte Carlo simulation for the arithmetic average reset. Cheng and Zhang (2000) used both historical stock prices and an approximated geometric closed-form solution to analyze general reset options with different reset dates. Whereas Liao and Wang (2002) applied a martingale method with closed-form solutions for single-barrier options and conducted a Monte Carlo simulation using the closed-form solution as the control variate to price an arithmetic average reset option. Dai et al. (2005) provided analytic formula for geometric average trigger reset options. They suggest that pricing the arithmetic average trigger reset option can benefit from using the formula provided as the control variate in the Monte Carlo simulation.

Within the extensive literature on pricing options, comparatively little research has focused on the valuation of Asian reset options. In the past, we were unable to evaluate the Asian option presented in the closed-form solution facilely until Milesky and Posner (1998) provided a more accurate closed-form model. They proved that the limiting probability distribution of the sum of the log-normal variables follows the Reciprocal Gamma Distribution. Under the risk-neutral hypothesis, the value of the Asian option is equivalent to the present value of the expected value of future cash flow at maturity, thus the expected value is derived from the Reciprocal Gamma Distribution. Other scholars such as Turnbull and Wakeman (1991), Levy (1992) and Vorst (1992) also derived similar closed-form models with enough high accuracy. As for pricing methods in reset options, Cheng and Zhang (2000) obtained a closed-form pricing formula for a reset option with a discrete multiple reset dates in terms of the multivariate normal distribution under the risk-neutral framework. They characterized the reset options whose strike price will be reset to the prevailing stock price if the option is out-of-money. Additionally, Liao and Wang (2003) built on the work of Cheng and Zhang (2000) and evaluated the reset options with multiple strike resets and reset dates.

The lattice model has also been commonly used in pricing reset options. Boyle and Lau (1994) provided a flexible and well-accepted approach for pricing options. Generally, lattice models can converge quickly to the closed-form solution. Hsueh and Liu (2001) proposed a new design of reset options in which the option’s exercise price adjusts gradually, based on the amount of time the underlying spent beyond pre-specified reset levels. The diagram represents different possible-paths that might be followed by the stock price over the duration of an option. Dai et al. (2004) considered options that allow the holders to reset the strike price with a preset number of times at any moment during the life of the option. The Asian reset option can also be derived from lattice models. For example, Hull and White (1993) developed an extended binomial method in which they construct a tree with a vector of average prices stacked at each node. Conditioning on the current value of the average price, they apply a standard recursive valuation method for each level of average prices in the chosen vector of average prices. Chang et al. (2004) extended the algorithm of Hull and White (1993) and formulated a numerical approach to the arithmetic average reset option; however it lacked convergence guarantees. These studies have been critically important in laying the groundwork for understanding how to price reset options.

A few studies have focused on deriving the approximation for the arithmetic average reset option via the geometric average reset option. Cheng and Zhang (2000) analysed that the geometric average of stock prices can be scaled to approximate the arithmetic average of stock prices in order to derive and approximate a pricing formula for the arithmetic average
option price. In Kao and Lyuu (2003), they gave practical algorithms which converge quickly to the correct value as verified by simulation for pricing a moving-average-lookback option and a moving-average-reset-option under the stock market of Taiwan’s exchange. Furthermore, Dai et al. (2005) also mentioned that throughout numerical experiments, the pricing formula can be used to approximate the values of arithmetic average reset options accurately. The practically of the proposed methodology is demonstrated through algorithm.

This study may be critically important in laying the groundwork for understanding how to price arithmetic average reset options using the Green function method. In this paper, we develop the price of arithmetic average reset options as the solution to a reset problem of the Black-Scholes equation. The problem is then transformed into an initial value problem of the heat equation. The solution can be formulated explicitly via integration. Finally, the numerical method is used to calculate the price of the arithmetic average reset option. Two series of examples are provided. The first series demonstrates the influence of each variable on the arithmetic average reset option, while the second series shows that the difference between the geometric average reset option and arithmetic reset option is insignificant. The paper is organized as follows. The arithmetic average reset option is formally defined in section 2. Then, the PDE preliminaries and initial conditions of the reset option will be demonstrated, as well as the derivation of the payoff of the reset option via the algorithm. Afterward, section 3 illustrates two series of numerical examples. And finally, in the last section, conclusions and discussions will be raised.

2. Methodology

An arithmetic average reset option is an Asian reset option with the reset threshold derived from the arithmetic average of the underlying stock prices.

2.1 Arithmetic average reset options

We consider a European-style arithmetic average reset call option with an original strike price $E$ and a reset threshold $\alpha E$, where $\alpha$ is the reset rate. In addition, we discuss the provision of a European-style Asian reset call option with two pre-specified monitoring dates. During the life of the contract, we assume that the right of reset can be executed at two predetermined monitoring dates, $t_1, t_2$, with stock price $s_1$ and $s_2$ respectively, and $0 < t_1 < t_2 < T$, where $T$ is the expiration date. In regards to the average price, we take the arithmetic average stock price before the reset date, and show it as the average stock price at the $i^{th}$ reset date ($i=1, 2$). We present the time and corresponding stock price about this reset monitoring window in Figure 1.

![Figure 1. Time line](image)

Suppose that the underlying stock has no dividend through the entire period. In addition, the strike price of the option is able to be reset at the specific percentage of the closing stock price on the reset date $t_2$ if the average stock price $\bar{s}$ triggers the reset threshold $\alpha E$, where $\alpha$ is
the reset rate, $0 < \alpha < 1$, and $E$ is the original exercise price. In our study, the arithmetic average of the stock price is $\bar{s} = \frac{s_1 + s_2}{2}$. Hence, the terminal value of the reset option is

$$c(s, T) = \begin{cases} s - E^*, & \text{if } s \geq E^* \\ 0, & \text{if } s < E^* \end{cases},$$

(1)

where the strike price $E^*$ is as follows.

$$E^* = \begin{cases} \alpha E, & \text{if } \bar{s} \leq \alpha E \\ E, & \text{if } \bar{s} > \alpha E \end{cases}.$$  

(2)

That is, if no reset happens at the reset date $t_2$, the terminal payoff will be $\max\{s - E, 0\}$. Otherwise the terminal payoff will be $\max\{s - \alpha E, 0\}$, in which $\alpha E$ must follow the restrict rule if any reset trigger resets on the reset date.

2.2 PDE preliminaries and initial conditions

The Asian reset option is derived from the Black-Scholes partial differential equation (PDE). The stock price behavior follows the Geometric Brownian Motion (GBM) with an expected return $\mu$ and a volatility of stock price $\sigma$. From the Efficient Market Hypothesis (EMH), we often state that the underlying stock price $s$ must move randomly. The stock price can be described as the stochastic differential equation as follows.

$$ds = \mu s dt + \sigma s dz,$$

(3)

where $dz$ follows the standard Wiener Process. Under the risk-neutral hypothesis, the expected return $\mu$ must equal the risk-free interest rate $r_f$. At time $t$, for an arbitrary initial stock value $s_0$, the random variable $\ln\left(\frac{s}{s_0}\right)$ is normally distributed with mean $(r_f - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. That is,

$$\ln\left(\frac{s}{s_0}\right) \sim N\left((r_f - \frac{\sigma^2}{2})t, \sigma^2 t\right),$$  

(4)

where $N(\alpha, \beta)$ is the normal distribution function with mean $\alpha$ and variance $\beta$, and $\ln(.)$ is the logarithm to the base $e$, where $e$ is an irrational constant. Specifically, in equation (4), $(r_f - \frac{\sigma^2}{2})$ is the drift term of the stock price under the GBM.

Based on the Black-Scholes assumptions, the option price $c(s, t)$ is the function of the underlying asset price $s$ and the time $t$, and must satisfy the forward-type Black-Scholes equation:

$$\frac{\sigma^2}{2} s^2 C_{ss}(s, t) + r_f s C_s(s, t) + C_t(s, t) = r_f c(s, t),$$

(5)

where $\sigma$ denotes the volatility of the stock price, $r_f$ represents the risk-free interest rate. After we transform the time $t$ into the time-to-maturity $\tau = T - t$, the corresponding monitoring dates $t_1$ and $t_2$ will become $\tau_1$, $\tau_2$ respectively, and the option price $c(s, t)$ will become

$$C(s, \tau) = c(s, T - \tau).$$

(6)

Simultaneously, the forward-type Black Scholes equation (5) will be transformed to the backward-type equation as follows,

$$\frac{\sigma^2}{2} s^2 C_{ss}(s, \tau) + r_f s C_s(s, \tau) - C_\tau(s, \tau) = r_f C(s, \tau),$$

(7)
We can present the backward and forward styles of this reset monitoring window in Figure 2:

<table>
<thead>
<tr>
<th>Style</th>
<th>Time Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward-type</td>
<td>( t_0 ) = T - t_0 \quad t_1 = T - t_1 \quad t_2 = T - t_2 \quad \tau_T = T - T = 0</td>
</tr>
<tr>
<td>Backward-type</td>
<td>( \tau_0 = T - \tau_0 ) \quad \tau_1 = T - \tau_1 \quad \tau_2 = T - \tau_2 \quad \tau_T = T - T = 0</td>
</tr>
</tbody>
</table>

Figure 2. The corresponding time variables in forward-type and backward-type

2.3 Algorithm

In this subsection, we use the convolution of the initial data and the recursive integral method to solve the valuation of arithmetic average reset options. The valuations of the two states at the reset date \( \tau_2 \) is the initial data of the solution at monitoring date \( \tau_1 \) and the solution at the monitoring date \( \tau_1 \) is the convolution of initial data and the Green function. Since there is no boundary constraints imposed to the Asian reset option, the option valuation is a series of initial value problems.

If the stock price \( s_2 \) at time \( t_2 \) falls below this floating barrier price, which is dependent on the stock price \( s_1 \), we must adjust the strike price as the pre-set strike level \( \alpha E \). On the contrary, if the average value of underlying stock prices does not fall below the reset threshold, the reset option will remain not reset. Therefore, we consider two situations at the reset date \( \tau_2 \), \( C_2(s_2, \tau_2) \) and \( C_2^*(s_2, \tau_2) \) denote the option price with the strike prices \( E \) and \( \alpha E \) respectively. The integral representation of \( C_2(s_2, \tau_2) \) and \( C_2^*(s_2, \tau_2) \) are as follows.

\[
C_2(s_2, \tau_2) = \int_0^{\tau_2} (s-\alpha E)G(s,0; s_2, \tau_2)ds, \quad (8)
\]

\[
C_2^*(s_2, \tau_2) = \int_0^{\tau_2} (s-E)G(s,0; s_2, \tau_2)ds, \quad (9)
\]

where the transformed Green's function is:

\[
G(s, \tau; s_2, \tau_2) = \frac{1}{s} e^{-\tau_2 (\tau - \tau_2)} \exp \left[ -\frac{(\ln(s) + (r_f - \frac{\sigma^2}{2})(\tau - \tau_2))^2}{2\sigma^2(\tau_2 - \tau)} \right] H(\tau_2 - \tau), \quad (10)
\]

\( H(\tau_2 - \tau) \) is the Heviside step function,

\[
H(\tau_2 - \tau) = \begin{cases} 
1 & \text{if } \tau_2 - \tau > 0 \\
0 & \text{if } \tau_2 - \tau \leq 0 
\end{cases} \quad (11)
\]

Then, we evaluate the valuation option at monitoring date \( \tau_1 \) via the integral representation at \( \tau_2 \) as the initial data. We can combine the payoff of the two states with a reset threshold \( s_2^*(s_1) \), where \( s_2^*(s_1) \) is the reset threshold function of \( s_1 \), and valuate the option’s value at the first monitoring time through the Green function:

\[
C(s_1, \tau_1) = \int_0^{\tau_1} C_2^*(s_2, \tau_2)G(s_2, \tau_2; s_1, \tau_1)ds_2 + \int_{\tau_1}^\tau C_2(s_2, \tau_2)G(s_2, \tau_2; s_1, \tau_1)ds_2, \quad (12)
\]

When we set the stock price \( s_1 \) at time \( \tau_1 \), we can obtain a known reset threshold \( s_2^*(s_1) \). In valuing an arithmetic average reset option, the reset threshold function \( s_2^*(s_1) = 2\alpha E - s_1 \). The valuation of the arithmetic average reset option can be obtained as the below integral representation:
Substituting equation (12) into equation (13), we can derive:

\[ C(s_0, \tau_0) = \int_0^\infty C_1(s_1, \tau_1) G(s_1, \tau_1; s_0, \tau_0) ds_1. \]  

(13)

\[ C(s_0, \tau_0) = \int_0^\infty C_1(s_1, \tau_1) G(s_1, \tau_1; s_0, \tau_0) ds_1 
= \int_0^\infty \left( \int_0^{\tau_1(s_1)} C_1^*(s_2, \tau_2) G(s_2, \tau_2; s_1, \tau_1) ds_2 \right) G(s_1, \tau_1; s_0, \tau_0) ds_1 \]  

(14)

Notice that under this algorithm, the reset threshold function \( s_2^*(s_1) \) for pricing geometric average reset options will be \( s_2^*(s_1) = \frac{(aoE)^2}{n_1} \). Instruments that cover Asian and reset features, depending on the arithmetic average closing underlying price, lack a simple closed-form solution and are more difficult to price. Therefore, we use a numerical method to derive the outcomes.

3. Numerical examples

In this section, we discuss the convergence of our method and provide two series of numerical examples. The first part focuses on the sensitivity analysis of the arithmetic average reset options. In each case, we take a control parameter to examine the relationship among the prices of the reset options with different scenarios. The second part of numerical examples illustrates the impact on different parameters of the option and further examines the absolute differences between the arithmetic average and the geometric average reset options.

3.1 Convergence

We provide some numerical examples to discuss the performance of the recursive algorithm for different sets of parameters. For the European-style arithmetic average reset call options, we have an analytic integral representation of the option value. However, the integral function can be evaluated analytically only for special cases (such as, the European plain vanilla call or put options). For a complex reset clause, it must be evaluated numerically. Thus, we apply the convolution of the initial data and the Green function in the program of Microsoft Visual C++ and the Simpson’s rule to solve the difficulty of pricing.

Table 1 shows the convergence of the recursive integral method applying Simpson’s rule in the numerical integration process. We consider three scenarios, A, B, and C which are: in-the-money, at-the-money, and out-of-the-money respectively, and we set the volatility \( \sigma = 0.3 \), the reset rate \( \alpha = 0.7 \) and the risk-free rate \( r_f = 0.1 \). Since Simpson’s rule must fit a function with a pair of sub-intervals, we let each tested number of step nodes be an even number. In scenario A, the relative errors between the numerical solution of the integral method and the number of 1024 step nodes has reached an eight digital level for \( n = 256 \) case. Additionally, for scenario B and C, the relative errors for \( n = 256 \) cases even reached to a ten digital level, which shows that this method can lead to a high accuracy. From the fifth column of Table 1, we can conclude that the relative errors decrease rapidly as the number of step nodes increase. In addition to evaluating the validity of this approach, we also analyze the numerical order of convergence. The numerical order \( O(n) \) is a function of the node spaces \( n \) defined as follows:

\[ O(n) = \frac{\ln\left(\frac{\text{error}(n_{2i})}{\text{error}(n_i)}\right)}{\ln\left(\frac{n_{2i}}{n_i}\right)} = \frac{\ln(\text{error}(n_{2i}) - \text{error}(n_i))}{\ln(n_{2i} - n_i)}, \]  

(15)
where error($n_{2i}$) and error($n_i$) are the node spaces for $n=2i$ and $i$ respectively.

Table 1. The convergence of pricing arithmetic average reset options with different step nodes

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>$n$</th>
<th>$C_n$</th>
<th>$C_{1024} - C_n$</th>
<th>$(C_{1024} - C_n) / C_{1024}$ (%)</th>
<th>Numerical Order</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scenario A:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>in-the-money</td>
<td>64</td>
<td>6.37128725322</td>
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<td>0.0000048553</td>
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<tr>
<td></td>
<td>128</td>
<td>6.37128435261</td>
<td>-0.00000019285</td>
<td>0.0000003027</td>
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<td>6.37128417175</td>
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<td></td>
<td>512</td>
<td>6.37128416046</td>
<td>-0.000000000070</td>
<td>0.000000011</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>1024</td>
<td>6.37128415975</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td><strong>Scenario B:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>at-the-money</td>
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<td>16.77098467998</td>
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<tr>
<td></td>
<td>1024</td>
<td>16.77098472154</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td><strong>Scenario C:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>out-of-the-money</td>
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<tr>
<td></td>
<td>1024</td>
<td>32.40712945879</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

Note: $n$ is number of step nodes. The order of the different step nodes ($n$) are converged at a stable value, 4. The scenario A, B, and C represent the difference with strike price $E=80$, 100 and 120 respectively, while the volatility $\sigma=0.3$, the reset rate $\alpha=0.7$ and the risk-free rate $r_f=0.1$. Furthermore, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$, and the time-to-maturity $\tau_0=1$.

When the number of step nodes doubles in size, the error becomes approximately $\frac{1}{16}$ in size. That is, the numerical order is about 4 (as shown in Table 1), which reveals the stability of this method. In addition, in Figure 3, we illustrate the slope correspondence with each scenario according to equation 15. The line of each figure shows a line with a slope of 4 and passes the origin. Hence, we conclude that this method leads to a numerical order of convergence with the features of both accurate and stable.
We discuss the properties of the arithmetic average reset options in this subsection. Consider a reset call option with one-year maturity and an initial strike price of 100, where we set the reset rate $\alpha=0.7$, the risk-free interest rate $r_f=0.1$, and volatility $\sigma=0.3$. The strike price will be adjusted if the arithmetic average of underlying stock price falls below the reset threshold, $\alpha E$. We compare the prices of the arithmetic average reset option with different scenarios via changing a single parameter in each scenario. The results are shown in Figure 4 through Figure 8.

The valuations of arithmetic average reset option with various current underlying stock prices under different level of reset rates $\alpha$, $\alpha=0.6$, 0.7, and 0.8, are shown in Figure 4. Since the lower reset rate is less possible to reach compared with higher ones under the same conditions, it appears that the curves fluctuate in the intervals of the stock prices slightly lower than the reset rate, and the smaller reset rate leads to a larger fluctuation in option prices. When the underlying stock prices are in a high level, we can see from the numerical examples in Figure 4 shows that any level of reset rates $\alpha$ has little impact on the option value. This results from the fact that a higher level of the underlying stock prices usually means that the reset option is unlikely to be reset.

Figure 5 demonstrates the valuations of arithmetic average reset option under different levels of strike prices $E$. We consider three levels of exercise prices, $E=90$, 100, and 110. The prices of the reset options are relatively closer when the underlying stock price falls below the reset threshold. As the underlying stock passes through the reset threshold $\alpha E$ (i.e. 70 in this case), the characteristics of the arithmetic average reset option is similar to the plain vanilla option. Essentially, the lower the exercise price is, the higher the call option value will be. Options with a lower strike price lead to a lower cost at maturity when exercised. This will also lead to a higher option price as well as the arithmetic average reset option. Therefore, there exists a negative relation between the strike price and the option price.
Figure 4. The valuation of the arithmetic average reset option with various current underlying stock price under different reset rates $\alpha$, $\alpha=0.6$, 0.7, and 0.8. This graph shows the change in option price under different reset rates. In this example, we hold all the other variables constant, i.e., $E=100$, $r_f=0.1$, $\sigma=0.3$, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$, and the time-to-maturity $\tau_0=1$

Figure 5. The valuation of the arithmetic average reset option with various current underlying stock prices under different exercise prices $E$. This graph shows the change in option price under different exercise prices. In this example, we hold all the other variables constant, i.e., $r_f=0.1$, $\alpha=0.7$, $\sigma=0.3$, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$, and the time-to-maturity $\tau_0=1$. The lower exercise price leads to a higher option price since its cost at maturity is higher

Figure 6 shows the pricing of an arithmetic average reset option with various volatilities. We consider three levels of volatilities, $\sigma=10\%$, 20%, and 30%. The higher volatility means that there is a the higher probability of the underlying stock price triggering the reset right, therefore the curve of the reset option shows less fluctuation; whereas, a reset option with a lower volatility, as seen in Figure 6, is more sensitive to the underlying stock price.
The valuation of the arithmetic average reset option with various current underlying stock prices under different volatilities $\sigma$. This graph shows the change in option price under different volatilities. In this example, we hold all the other variables constant, i.e., $E=100$, $\alpha=0.7$, $r_f=0.1$, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$, and the time-to-maturity $\tau_0=1$. The lower volatility, the reset option is more sensitive, which fluctuates near the reset price, $aE$.

The valuation of an arithmetic average reset option under different risk-free interest rates $r_f$ is schematized in Figure 7, while the pricing of arithmetic average reset option under different time-to-maturity $\tau_0$’s is given in Figure 8. We consider three levels of risk-free interest rates, $r_f=0.1, 0.2, \text{ and } 0.3$ in Figure 7, and three different time-to-maturities (in year), $\tau_0=0.25, 0.5, \text{ and } 1$ in Figure 8. The prices of the options are positively correlated to the risk-free interest rates and the time-to-maturity of the option. Both $r_f$ and $\tau_0$ reveal similar characteristics of the plain vanilla option. Under the GBM, with the higher level of risk-free interest rate $r_f$, the drift term of the underlying stock price will rise as well. That is, the probability of growth in the future stock price increases as the risk-free interest rate $r_f$ increases, which will ultimately lead to a higher value of option price. Meanwhile, this is also reflected in the higher prices of the options with longer maturity. Since $\sigma\sqrt{\tau_0}$, the standard deviation of the underlying stock price is directly proportion to the time-to-maturity $\tau_0$; a higher value of $\tau_0$ will lead to a higher value of $\sigma\sqrt{\tau_0}$.

Figure 6. The valuation of the arithmetic average reset option with various current underlying stock prices under different risk-free interest rates $r_f$ is schematized in Figure 7, while the pricing of arithmetic average reset option under different time-to-maturity $\tau_0$’s is given in Figure 8. We consider three levels of risk-free interest rates, $r_f=0.1, 0.2, \text{ and } 0.3$ in Figure 7, and three different time-to-maturities (in year), $\tau_0=0.25, 0.5, \text{ and } 1$ in Figure 8. The prices of the options are positively correlated to the risk-free interest rates and the time-to-maturity of the option. Both $r_f$ and $\tau_0$ reveal similar characteristics of the plain vanilla option. Under the GBM, with the higher level of risk-free interest rate $r_f$, the drift term of the underlying stock price will rise as well. That is, the probability of growth in the future stock price increases as the risk-free interest rate $r_f$ increases, which will ultimately lead to a higher value of option price. Meanwhile, this is also reflected in the higher prices of the options with longer maturity. Since $\sigma\sqrt{\tau_0}$, the standard deviation of the underlying stock price is directly proportion to the time-to-maturity $\tau_0$; a higher value of $\tau_0$ will lead to a higher value of $\sigma\sqrt{\tau_0}$.

Figure 7. The valuation of the arithmetic average reset option with various current underlying stock prices under different risk-free interest rates $r_f$. This graph shows the change in option price under different risk-free rates. The other variables constant are: $E=100$, $\alpha=0.7$, $\sigma=0.3$, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$ and the time-to-maturity $\tau_0=1$. The higher risk-free rate leads to a higher option price.
Additionally, due to the fact that the maximum loss for a call option is the premium whereas the maximum profit has no certain boundary, the price of the option will rise in this case. Hence, under these two parameters, the price of the plain vanilla option increases as well as the arithmetic average reset option.

Figure 8. The valuation of the arithmetic average reset option with various current underlying stock price under different time-to-maturities $\tau_0=1$. This graph shows the change in option price under different time-to-maturities. The other variables constant are: $E=100$, $\alpha=0.7$, $\sigma=0.3$, $r_f=0.1$. The monitoring dates $\tau_1=0.5$, $\tau_2=0.75$. The longer time-to-maturity leads to a smoother and higher option price.

3.3 Difference in arithmetic average and geometric average reset option

In the research of Cheng and Zhang (2000) and Dai, Fang, and Lyuu (2005) both stated that the difference between an arithmetic average and a geometric average reset options is very small. Thus they use the geometric average to approximate the arithmetic average. In this subsection, we provide some numerical examples to examine the absolute differences between the arithmetic average and the geometric average reset options under different scenarios.

We investigate the difference between the arithmetic average and the geometric average reset options with the consecutive and discrete reset monitoring clauses, respectively. For Table 2, we consider a one-year maturity arithmetic average reset call option with the initial strike price 100, the volatility of the underlying stock price 0.3, and the risk-free interest rate 0.1. Under our contract, when the average underlying asset price falls below 70% of the original strike price, the reset right will be triggered. The reset threshold for arithmetic average and the geometric average reset options can be interpreted as $\bar{s}^* = \frac{E - \sigma}{\alpha + \frac{1}{2}}$ and $\bar{s}^* = \sqrt{s_1 s_2}$, respectively. We take the absolute difference of the two pricing algorithms under the same parameters. In Table 2, $RE$ represents the relative error and is defined as follows:

$$RE = \frac{|A\text{-reset} - G\text{-reset}|}{G\text{-reset}}.$$  \hspace{5cm} (16)

where $A\text{-reset}$ and $G\text{-reset}$ represent the value of the arithmetic average and the geometric average reset options respectively. We choose two consecutive monitoring dates, $\tau_1^* = 0.5$ and $\tau_2^* = 0.503$ in the panel A of Table 2, as well as two discrete monitoring dates, $\tau_1=0.5$, $\tau_2=0.75$ (3-month period) in the panel B of Table 2. We see that the results of the maximum relative errors for two average algorithms are 0.000132 and 0.0164 showed in the panel A and B of Table 2, respectively. This means the differences between two average algorithms is very small in the case of consecutive and discrete reset monitoring clauses respectively.
Table 2. The difference of the valuation between arithmetic average and geometric average reset options in consecutive and discrete reset monitoring clauses respectively.

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>Panel A: two consecutive monitoring clauses</th>
<th>Panel B: two discrete monitoring clauses</th>
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<tr>
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<td>A-reset</td>
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<tr>
<td>120</td>
<td>32.42389185</td>
<td>32.42388148</td>
</tr>
</tbody>
</table>

Note: The column of A-reset represents the valuation of arithmetic average reset option, while the column of G-reset represents the valuation of geometric average reset option. The column of Difference means the absolute difference between A-reset and G-reset. $\alpha=0.7$, $\sigma=0.3$. The volatility $\sigma=0.3$, the reset rate $\alpha=0.7$ and the risk-free rate $\kappa=0.1$. The monitoring dates are 0.5 and 0.503, and the time-to-maturity $\tau=1$. 
Figure 9 through Figure 13 illustrate the sensitivity analysis of the absolute differences between the values of arithmetic average and geometric average reset options. All the curves in Figure 9 through Figure 13 are exhibited in bell shapes.

In Figures 9 and 10, we demonstrate the differences in the geometric average and the arithmetic average reset options under different levels of reset rates, \( \alpha = 0.6, 0.7, 0.8 \), and different strike prices, \( E = 90, 100, 110 \), respectively. The reset threshold for this option contract is \( \alpha E \), thus for the reset rate \( \alpha \) and the original strike price \( E \), holding either of them constant, the other will result in altering the differences between the two reset clauses. Hence, Figures 9 and 10 show similar patterns of influence on the price of the reset option. Each curve results in having the peak in the interval below its corresponding reset criteria, \( \alpha E \).

![Figure 9](image1)

Figure 9. The difference of the valuation between arithmetic-average and geometric-average reset options under different reset rates \( \alpha \). The solid, dashes and dot-dashed lines represent the difference with reset rates, \( \alpha = 0.6, 0.7 \) and 0.8 respectively. In this example the original strike price \( E = 100 \), while the volatility \( \sigma = 0.3 \) and the risk-free rate \( r_f = 0.1 \). Furthermore, the monitoring dates \( \tau_1 = 0.5, \tau_2 = 0.75 \), and the time-to-maturity \( \tau_0 = 1 \).

![Figure 10](image2)

Figure 10. The difference of the valuation between arithmetic-average and geometric-average reset options under different exercise prices \( E \). The solid, dashes and dot-dashed lines represent the difference with strike prices 90, 100 and 110 respectively. In this example the original volatility \( \sigma = 0.3 \), while the reset rate \( \alpha = 0.7 \) and the risk-free rate \( r_f = 0.1 \). Furthermore, the monitoring dates \( \tau_1 = 0.5, \tau_2 = 0.75 \), and the time-to-maturity \( \tau_0 = 1 \).
In Figure 11, we demonstrate the absolute differences in a geometric average and an arithmetic average reset option under different volatilities, $\sigma = 10\%, 20\%$, and $30\%$, while in Figure 12, we observe the differences in a geometric average and an arithmetic average reset option under different time-to-maturities, $\tau_0 = 0.25, 0.5,$ and $1$ (in year). Since the standard deviation of the underlying stock price over a period of time $T$ is $\sigma \sqrt{\tau_0}$, where $\sigma$ is the underlying volatility of the underlying stock price, either changing the volatility $\sigma$ or the time-to-maturity $\tau_0$ will both have positive impact on the standard deviation $\sigma \sqrt{\tau_0}$. Additionally, a higher level of the volatility $\sigma$ of the time-to-maturity $\tau_0$ will results in a larger value of the absolute difference between an arithmetic average and a geometric average reset options. Despite the covered bell shape area that shows the difference in pricing reset options under the arithmetic average and geometric average, the y-axis, Difference in A-reset and G-reset options, showed in both figures schematize the insignificant differences (approximate to 0.08 in Figure 11 and approximate to 0.05 in Figure 12).

![Figure 11](image-url)

Figure 11. The difference of the valuation between arithmetic-average and geometric-average reset options under different volatilities $\sigma$. The solid, dashes and dot-dashes lines represent the difference with volatilities 0.1, 0.3 and 0.5 respectively. In this example the original strike price $E = 100$, while the reset rate $\alpha = 0.7$ and the risk-free rate $r_f = 0.1$. Furthermore, the monitoring dates $\tau_1 = 0.5$, $\tau_2 = 0.75$, and the time-to-maturity $\tau_0 = 1$

Figure 13 shows the absolute differences in a geometric average and an arithmetic average reset option under different risk-free rates, $r_f = 0.1$, 0.2, and 0.3, respectively. The peaks all occur in the interval below the reset criteria, and $r_f = 0.3$ leads to a relatively higher difference in option value in this case. Essentially, the stock prices under the reset threshold $aE$ show comparatively significant differences between the arithmetic average and geometric average reset options under different risk-free interest rates $r_f$. 
The Current Stock Price Difference in A-reset and G-reset Options

Figure 12. The difference of the valuation between arithmetic-average and geometric-average reset options under different time-to-maturities $\tau_0$. The solid, dashes and dot-dashed lines represent the difference with expiration 0.25, 0.5 and 1 respectively. In this example the original strike price $E=100$, while the reset rate $\alpha=0.7$, the risk-free rate $r_f=0.1$, and the volatility $\sigma=0.3$.

Figure 13. The difference of the valuation between arithmetic-average and geometric-average reset options under different risk-free interest rates $r_f$. The solid, dashes and dot-dashed lines represent the difference with $r_f=0.1$, 0.2 and 0.3 respectively. In this example the original strike price $E=100$, while the reset rate $\alpha=0.7$ and the volatility $\sigma=0.3$. Furthermore, the monitoring dates $\tau_1=0.5$, $\tau_2=0.75$, and the time-to-maturity $\tau_0=1$.

4. Conclusions

The reset feature embedded in an option entitles the holder the right to reset a certain term under the option contract. This may be interpreted as the privilege given to the holder to convert the original option into a new one via changing the strike price. This study exhibits the valuation of European-style Asian reset call options with the arithmetic average closing price as a comparison with a pre-specified reset level over each discrete monitoring window. We use both the convolution of the initial data and the Green function via the recursive integral method to solve the Black-Scholes equation and further valuate the intrinsic prices of
reset options according to the reset option contract. In this paper, we present the results of the valuation of the arithmetic average reset option. The more likely explanation resets in the nature of pricing vanilla options. Through our method, the numerical order approximately equals 4, which shows that this method is stable and efficient in the numerical order of convergence. We also discuss the absolute differences of the arithmetic average and geometric average reset options to see the particular characteristics and reset value caused from the reset clause of an Asian reset option. These findings are in accord with the previous studies of Cheng and Zhang (2000) and Dai, Fang, and Lyuu (2005), despite the fact that these studies used very different measures of valuation. Using the method provided in this paper, we can evaluate the arithmetic average and the geometric average reset options via changing the algorithm. The results show that the absolute differences are exhibited in bell shapes and the values are relatively small compared to the arithmetic average reset option as a standard.

References


