Pricing and Hedging Strategy for Options with Default and Liquidity Risk

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Abstract

This study applies fuzzy set theory to the vulnerable Black-Scholes (1973) or Merton (1973) formula. Expectations of heterogeneity mean option prices are expected to be imprecise, thus making it natural to consider fuzziness to handle this. This article presents a fuzzy approach to value Black-Scholes options subject to non-identical rationality and correlated credit risk. Although no analytical solution is available, this study employs a fuzzy approach to derive an approximate analytical expression for the upper and lower bounds of the European fuzzy vulnerable option price. Furthermore, the Greeks and hedging strategy of the proposed model are also provided in this article.

Keywords: Fuzzy measure theory, vulnerable option, liquidity risk, credit risk, non-identical rationality

1. Introduction

Numerous banks and dealers actively trade derivatives with their customers, and the volume of over-the-counter (OTC) warrants recently has grown rapidly. Without the margin and rule requirements associated with an exchange house, the writers of such instruments have both the motivation and opportunity to default. Default risk is not negligible for investors trading OTC options, since holders are exposed to credit risk potential losses. To cover potential losses, holders reduce option price to obtain a credit risk premium. It is important to account for this credit risk premium when pricing OTC options and the model must consider credit risk when the probability of default is high.


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collateral, technology and so on, are difficult to be traded or even non-tradable. Since the assets are not sufficiently liquid, liquidity problems increase option default probability and force writers to reduce the option price to compensate investors. Thus liquidity risk premium is another important determinant of the price of options with correlated credit risk.

Cherubini and Della Lunga (2001) use fuzzy measure theory to simultaneously solve the liquidity risk and credit risk problems. However, their work mainly focuses on the liquidity risk of the underlying asset, claiming that illiquidity can spill over to derivatives and increase the default risk of the latter. That is, pricing a defaultable security implies evaluating an illiquid option. Han and Zheng (2005) follow Cherubini and Della Lunga (2001) and apply fuzzy options to analyze credit risk for Chinese municipal bonds. Han and Zheng (2005) use the \( \lambda \)-additive fuzzy measures to relax the “identical rationality” assumption. In classical economic theory identical rationality is an ideal assumption, which assumes that there is a representative market participant in the market. The information, preferences, expectations and the investment strategy of this representative participant represent all the behavior of all market participants. The option pricing model, such as Black and Scholes formula which follows the assumption of identical rationality, derives the unique option price from the unique probability distribution of the uncertain future payoff. Hence, all investors share the same market information and take the same investment strategy under identical rationality assumption. This phenomenon departs from the actual market condition that each investor has it own expectation about future market condition and takes different trading strategy. In other words, no trade will occur if all investors take the same trading strategy, and thus investment behavior imprecision comes from heterogeneous investor expectations. Han and Zheng (2005) consider the problem of imprecision when pricing bonds. Cherubini and Della Lunga (2001) and Han and Zheng (2005) both handle credit risk in different fields, and do not consider credit risk to be directly induced from option writer default. This study thus incorporates credit risk, liquidity risk and imprecise investment behavior when pricing options, combining the strengths of Klein (1996), Cherubini and Della Lunga (2001), and Han and Zheng (2005). The proposed model not only allows other liabilities in writer capital structure, as well as a correlation between writer and underlying assets, but also considers the non-tradable property of writer assets and investor non-identical rationality.

To achieve above aim, this study uses a family of \( \lambda \)-additive fuzzy measures to price the vulnerable option model in Klein (1996), and then calculates the option price by taking the Choquet expectations under these measures. Individual investors with different risk preferences or expectation thus can have different expected option prices, even if the available information is homogeneous. The results of this pricing method are consistent with the situation of individual rationality. Since the pricing formula is in an integrated form, this study provides an approximate analytic solution, numerical option price results and hedge ratios.

Due to the non-homogeneous belief, the traditional classical measure should be generalized as the fuzzy measure by replacing the additivity axiom of the classical measure with weaker axioms of monotonicity and continuity. A probability measure, which is closely connected with classical measure, is a set function that assigns measure 0 to the empty set and a nonnegative number to any other set, and that is additive. Dempster (1967) was the first scholar to propose this theory which is based on two types of non-additive measure. Special types of lower and upper probabilities, referred to as belief measures and plausibility measures, were later introduced and thoroughly investigated by Shafer (1976). Another theory based on non-additive measures, fuzzy sets theory, was proposed by Zadeh (1965) and has since developed rapidly. According to the definition first given by Zadeh (1965), a fuzzy set describes an event set without clear boundaries. Additionally, Kwakernaak (1978) proposed the notion of fuzzy random variable and Puri and Ralescu (1986) then further developed the
same concept. A specific fuzzy measure, the $\lambda$-additive fuzzy measure, proposed by Cherubini (1997, 2001), is employed in this study.

A fuzzy integral is a general term for integrals with respect to the fuzzy measure, and several kinds exist, including the Choquet and Sugeno integrals. This study mainly applies the Choquet integral (expectation) to obtain the fuzzy option price. The default problem results from bond value pricing, and this study aims to handle this problem regarding option pricing. Two pricing approaches have been used to model credit risk. The first is the reduced-form approach and the second is the structural approach; this article focuses on the latter.

Research in psychology suggests that people are overconfident, with Alpert and Raiffa (1982), and Brenner et al. (1996) and other calibration studies found that people overestimate the precision of their knowledge. Camerer (1995) argued that even experts can display overconfidence, and Hirshleifer (2001) and Barber and Odean (2002) presented extensive reviews of the literature as it relates to investing. Overconfident investors believe more strongly in their own assessments of an asset’s value than those of others. In this way, overconfidence leads to heterogeneous beliefs or differences in opinions. According to behavioral finance theory, individuals make decisions guided by heuristics, or practical rules, thinking in a way which deviates from the statistic rules. Another view of the upper and lower bounds of the vulnerable option price is based on the fact that overconfident investors are always able to justify their subjective probabilities.

Besides the valuation of the vulnerable option under a heterogeneous belief of each market participant, risk management policies of the financial institutions which sell options or other derivative in the OTC markets are also very important. This study also derives the Greek letters of the fuzzy vulnerable option for the ease in dynamic hedging and suggests a hedging strategy when market participant faces underlying stock price risk and the risk of the trading counterparty under heterogeneity. This analysis is applicable to market makers in options on an exchange as well as to traders working for financial institution.

The remainder of this study is organized as follows. Section 2 describes fuzzy set theory. Section 3 first introduces a traditional method of pricing the vulnerable option model, as developed by Klein (1996), under the martingale pricing method, and then extends this to a fuzzified version of the vulnerable options formula. The $(\lambda, \lambda^*)$-interval of the fuzzy vulnerable option is then derived using a non-additive measure, specifically a sub-additive $\lambda$. Moreover, the lower bound of the expected value is calculated using the lower Choquet integral with respect to this measure, and the upper bound is obtained by calculating the corresponding upper Choquet integral and the hedging strategies, similar to the approach used by Black and Scholes (1973), which is also designed in this section. In Section 4, we first explain the advantage of using a $\lambda$-addictive fuzzy measure and then undertake sensitivity and numerical analyses of fuzzy vulnerable options. Finally we give an application to price employee stock options whose underlying asset is only thinly or not traded, while Section 5 summarizes conclusions of this work.

2. Fuzzy set theory

A fuzzy set is a set whose boundary is not required to be sharp. That is, the change from non-membership to membership is allowed to be gradual rather than abrupt. This gradual change is expressed by a membership function of the fuzzy set, which assigns to each individual of a given universal set its degree of membership in the fuzzy set. If these degrees are expressed by values in the unit interval $[0,1]$, the fuzzy set is called standard. Standard fuzzy sets as well as standard operations on fuzzy sets were introduced in the seminal paper by Zadeh (1965), and has since been applied in numerous scientific areas. Zadeh (1965) also developed a suite of algebra manipulation for fuzzy set and this algebra is good in handing the
decision-making process under imprecise information. This theory is applied by a lot of researchers in finance to deal with the investment decisions in an uncertain market condition.

Measuring fuzziness using the concept of probability is difficult (Zadeh, 1965), because probability is used to measure randomness, which is relevant to event occurrence, while fuzziness is relevant to event degree (Bellman and Zadeh, 1970). The membership function $\mu_{E}: X \rightarrow [0,1]$ expresses the fuzzy set $E$ in set $X$ where the element $x \in X$, when $\mu_{E}(x) \in [0,1]$. For example, $E$ may denote a set of red objects, while $X$ denotes a mixed set of red and non-red objects. The grade of membership 1 is traditionally assigned to objects which totally belong to $E$, namely they are red, while the membership value 0 is assigned to objects which are not red. Pink objects thus have membership grades lying between 0 and 1. The more an element or object $x$ belongs to $E$, the closer to 1 is its membership grade $\mu_{E}(x)$. Consequently, the membership grades reflect object ordering. This approach was first developed by Zadeh (1965) as a tool for modeling human-centered systems (Zadeh, 1973).

Several principal fuzzy integrals exist from which others are modified. This study mainly focuses on Choquet integral, because it is the most natural fuzzy integral, and derives the fuzzy option prices from the Choquet expectation.

Let $\Omega$ represent a nonempty set, and let $\Sigma$ denote $\sigma$-algebra on $\Omega$, while $\{\Omega, \Sigma\}$ is a measurable space. Some basic concepts are outlined here to introduce fuzzy set theory. The fuzzy measure is defined as follows.

**Definition 1: Fuzzy measure**

Given a measurable space $\{\Omega, \Sigma\}$, a finite monotone set function $\mu$ is a fuzzy measure if and only if:

(a) $\mu(\emptyset) = 0$.
(b) if $E_i, E_j \in \Sigma$ and $E_i \subset E_j$ then $\mu(E_i) \leq \mu(E_j)$.

In the context of some applications, it is desirable that monotone measures also satisfy one or both of the following requirements:

(c) Continuity from below

\[ \text{if } \{E_n\} \in \Sigma, E_1 \subset E_2 \subset \cdots \text{ and } \bigcup_{n=1}^{\infty} E_n \in \Sigma \text{ then } \lim_{n \to \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right). \]

(d) Continuity from above

\[ \text{if } \{E_n\} \in \Sigma, E_1 \supset E_2 \supset \cdots \text{ and } \bigcap_{n=1}^{\infty} E_n \in \Sigma \text{ then } \lim_{n \to \infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right). \]

If in addition it is $\mu(\Omega) = 1$, the fuzzy measure is called regular.

**Definition 2: $\lambda$-additive fuzzy measure**

A $\lambda$-additive fuzzy measure is defined on $\{\Omega, \Sigma\}$, denoted by $\mu$, which possesses the property that for arbitrary $l \in (-1, \infty)$, $E_i, E_j \in \Sigma$, $E_i \cap E_j = \emptyset$. 

\[ \mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j) + \lambda\mu(E_i)\mu(E_j). \]

Furthermore, when $E_n \in \Sigma (n = 1, 2, 3\ldots)$ are disjointed with each other, the following is obtained.

\[ \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \begin{cases} \sum_{n=1}^{\infty} \mu(E_n), & \lambda = 0 \\ \frac{1}{\lambda} \left\{ \prod_{n=1}^{\infty} \left[1 + \lambda \mu(E_n)\right]^{-1}\right\}, & \lambda \neq 0 \end{cases} \]

There is a correspondence between the $\lambda$-additive fuzzy measure and risk-neutral probability measure when $\lambda \neq 0$. Ye and Zhang (1997) defined the relationship between these
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two measures: If $Q$ is a risk-neutral probability measure, then $\nu = \frac{1}{\lambda} \left[ (1 + \lambda)^0 - 1 \right]$ is a $\lambda$-additive fuzzy measure, and if $\nu$ is a $\lambda$-additive fuzzy measure on $\Sigma$, then $Q = \ln(1 + \lambda(1 + \lambda
u))$ is a risk-neutral probability measure on $\Sigma$. In the financial market, this kind of non-identical rationality can be seen in the subjective probabilities assigned by investors to their own average weighted price, and this subjective probability can be transformed from an identical mathematical expression based on fuzzy measures. By choosing the parameter $\lambda$, each investor assigns their own subjective measure when pricing assets. According to Cherubini and Della Lunga (1997, 2001), this paper applied the $\lambda$-additive fuzzy measure to the asset pricing problem and found intervals for the derivative price.

**Definition 3: Duality of the $\lambda$-additive fuzzy measure**

In the $\lambda$-additive fuzzy measure, it is also easy to recover a duality result between the sub- and super-additive measures. Simply define $\lambda^*$ as:

$$\lambda^* = -\frac{\lambda}{1 + \lambda}$$

Then it can be verified that, for any subset $E$,

$$\mu^\lambda(E) + \mu^{\lambda^*}(E^C) = 1$$

where $E^C$ denotes the complement to any subset of $E$. This means that for any sub-additive measure, it is possible to construct a super-additive measure and the sum of the two measures over $E$ and $E^C$ equals 1. In other words, if $\mu^\lambda$ is an $\lambda$-additive measure on $\Sigma$, then $\mu^{\lambda^*}$ is an $\lambda^*$-additive measure on $\Sigma$.

When we apply this method to options pricing, it is necessary to learn how to perform integration in fuzzy measure theory. To conclude our description of the basic fuzzy measure equipment, we have to touch briefly on results on integration, i.e. how to take expectations in a fuzzy measure space. This article mainly discusses the Choquet integral, because it is the most natural fuzzy integral, and the fuzzy vulnerable option prices are thus derived from the Choquet expectation.

**Definition 4: Choquet integral**

While in fuzzy measure theory quite general concepts of integration, such as fuzzy integrals or pan-integrals are used, in this study, we stick to the concept of integration proposed by Choquet (1954), and are defined as follows. Letting $f(X(\omega))$ denote the non-negative measurable real function on $\Omega$ then the Choquet integral based on fuzzy measure $\mu$ is:

$$(c) \int f(X)d\mu = \int \mu(\omega \in \Omega : f(X) \geq x)dx$$

where the right-hand side is the Riemann integral and $(c)$ reads “Choquet integral” and $\mu$ is a not-necessarily additive measure. If the measure is additive, the integral coincides with the common mathematical expectation.

**3. The model**

In this section we show how the previous analysis can result in a very simple concrete application. A fuzzified version of the option pricing model with default risk is proposed in this article to resolve the drawback of works by Johnson and Stulz (1987), Hull and White (1995) and Klein (1996), who considered non-tradable assets, such as firm value or collateral assets as tradable ones. In addition, this study also adopts the idea of Cherubini (1997) and the pricing method of Han and Zheng (2005) to resort to a particular class of sub-additive measures, fuzzy measures, to compute the minimum value of a long (short) position in some
asset, where the minimum is taken on a properly defined set of probability measures. The idea is to evaluate the long (short) position using the lower (upper) Choquet integral with respect to a sub-additive function.

3.1 Klein’s (1996) model

Similar to the Black-Scholes framework, let $Q$ denote the risk-neutral probability measure, both the stochastic processes of the underlying stock price, $S_t$, and option writer’s assets value, $V_t$, are assumed to follow geometric Brownian motions:

\[
\frac{dS}{S} = rdt + \sigma_S dz^Q_S, \\
\frac{dV}{V} = rdt + \sigma_V dz^Q_V,
\]

where $r$ be the risk-free rate, and volatility $\sigma_S$ and $\sigma_V$ represent the instantaneous standard deviations of return on the assets underlying the option and those of the counterparty, respectively; furthermore $\rho_{SV}$ denotes the correlation between $S_t$ and $V_t$.

In the classical economic theory, all the market participants are assumed to be identical rationality. The vulnerable call option $v_{CT}$ and the vulnerable put option $v_{PT}$ prices can be considered as follows:

\[
v_{CT} = e^{-r(T-t)}E^Q_t[A_f], A_f = \left[\mathbb{M}(S_T - K, 0)\left[1|V_T \geq D^*\right] + \left[\frac{1-\beta V_T}{D^*}\right]1|V_T < D^*\right]\]

and

\[
v_{PT} = e^{-r(T-t)}E^Q_t[B_f], B_f = \left[\mathbb{M}(K - S_T, 0)\left[1|V_T \geq D^*\right] + \left[\frac{1-\beta V_T}{D^*}\right]1|V_T < D^*\right]\]

Where $D$ represents the total number of claims of the trading counterparty; Since there exists the probability that a counterparty continuing operations even while $V_t$ is less than $D$, $D^*$ may be less than $D$. $T - t$ is the time to maturity. $\beta$ is the deadweight costs as the trading counterparty goes bankruptcy and it is exhibited as a percentage of the value of the assets of the counterparty.

Theorem 1: (Klein, 1996)

The pricing formula of vulnerable call options is as follows:

\[
v_{CT} = S_tN_2\left(a_1, a_2, \rho_{SV}\right) - e^{-r(T-t)}KN_2\left(b_1, b_2, \rho_{SV}\right) + \frac{1-\beta V_T}{D^*}S_te^{r\rho_{SV}\sigma_V(T-t)}N_2\left(c_1, c_2, -\rho_{SV}\right) - KN_2\left(d_1, d_2, -\rho_{SV}\right)
\]

and

\[
a_1 = \frac{\ln\left(S_t\right) + (r + \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}} = b_1 + \sigma_S\sqrt{T-t}, \\
b_2 = \frac{\ln\left(V_T\right) + (r + \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)(T-t)}{\sigma_V\sqrt{T-t}} = b_2 + \rho_{SV}\sigma_S\sqrt{T-t}, \\
c_1 = b_1 + (\sigma_S + \rho_{SV}\sigma_V)\sqrt{T-t}, c_2 = \frac{\ln\left(V_T\right) + (r + \frac{1}{2}\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)(T-t)}{\sigma_V\sqrt{T-t}}, \\
d_1 = b_1 + \rho_{SV}\sigma_V\sqrt{T-t}, d_2 = -b_2 + \sigma_V\sqrt{T-t}.
\]

where $N_2$ is the cumulative bivariate normal distribution function.
The formula of the vulnerable put options can be derived in the same way. Even the Klein’s model has some advantages to represent credit risk modeling. For example, it allows the option writer to have all kind liabilities; it investigates the correlation between the underlying asset of the option and the collateral asset of trading counterparty; it suggests option holders only receive proportional nominal claims from option writer and this proportion is related to the residual value of the counterparty’s assets on the event of default. It still has one drawback, that is, the firm value of the option writer, \( V \), which is not traded in the real market, is assumed to be a tradable security in his model. Hence, the fuzzy measure theory is employed in this study to incorporate the liquidity or credit risks into option writer's assets value.

### 3.2 Fuzzy vulnerable options

This section demonstrates how to use the fuzzy measure approach to introduce liquidity or credit risk into Klein (1996). Vulnerable options are those on which the writer may default, while fuzzy vulnerable options are those that are subject to credit risk in a fuzzy environment. To simplify the comparison, the following discussion will use the same notation as the previous section.

**Theorem 2: Vulnerable options with each investor’s own subjective measure \( \lambda \)**

As all investors are non-identical rationality, the value of the call and the put options with default risk can be derived as follows.

\[
\begin{align*}
vc_i^\lambda &= e^{-r(T-t)}CE_i^{Q,\lambda} \left[ A_t \right] = e^{-r(T-t)}E_i^{Q,\lambda} \left[ A_t \right] , \quad \lambda = 0 \\
&\neq e^{-r(T-t)}E_i^{Q,\lambda} \left[ A_t \right] , \quad \lambda \neq 0
\end{align*}
\]

and

\[
\begin{align*}
vp_i^\lambda &= e^{-r(T-t)}CE_i^{Q,\lambda} \left[ B_t \right] = e^{-r(T-t)}E_i^{Q,\lambda} \left[ B_t \right] , \quad \lambda = 0 \\
&\neq e^{-r(T-t)}E_i^{Q,\lambda} \left[ B_t \right] , \quad \lambda \neq 0
\end{align*}
\]

where \( \lambda \) denotes each investor’s own subjective measure when evaluating vulnerable call \( vc_i \) and put \( vp_i \). \( CE_i^{Q,\lambda} \) denotes Choquet’s expectation under his/her own subjective measure.

Each investor has his/her own perceived price when trading in the financial market. Before he/she makes any investment decision, he/she would compare his/her own price \( vc_i^\lambda \) (\( vp_i^\lambda \)) with the market price realized as \( A_t \), which is the reference price. If his/her own perceived price is equal to realized market price, he/she will stay put and take no action. If his/her own perceived price is larger than the realized market price, he/she will long the asset. If his/her own perceived price is less than the realized market price, he/she will short it. The final market price can lie between the two perceived prices, the prices of the long and short positions, respectively. Hence, the final market price will be collectively expected by investors as an interval. As the numbers of investors are large enough in the market, each investor’s own perceived price may vary continuously, which will cause the final market price to become a fuzzy set.

The Choquet integral \( (c)\hat{\int} \) can be used to obtain fuzzy vulnerable call options with measure \( \lambda \):

\[
vc_i^{\lambda=0} = e^{-r(T-t)} \left\{ (c)\int (S_t - K)_+ 1_{(s_t,s_d)}(v_t,s_d) dv + (c)\int (S_t - K) \frac{(v_t,s_d)}{b} 1_{(s_t,s_d)}(v_t,s_d) dv \right\} 
\]

(5)

To facilitate calculation, this study uses an approximate expansion for a fuzzy vulnerable call option with measure \( \lambda \) (shown in the Appendix):
\[ v_{c_t}^{\lambda=0} = S_t \mu \left[ N_2 \left( a_1, a_2, \rho \right) \right] \left( 1 + \lambda \right) N_2 \left( b_1, b_2, \rho \right) - K e^{-\left( T-t \right)} \mu \left[ N_2 \left( b_1, b_2, \rho \right) \right] \\
+ \frac{1}{\bar{D}} \left[ S_t e^{+r_0 \sigma_0 \sigma_0 (T-t)} \mu \left[ N_2 \left( c_1, c_2, -\rho_{SV} \right) \right] \left( 1 + \lambda \right) N_2 \left( d_1, d_2, -\rho_{SV} \right) \right] \\
- K \mu \left[ N_2 \left( d_1, d_2, -\rho_{SV} \right) \right] \] 

where the relationship of the \( \lambda \)-additive fuzzy measure is

\[ \mu_\lambda (Q) = \begin{cases} \frac{Q}{1 + \left( 1 + \lambda \right)^Q - 1}, & \lambda \neq 0 \\ \frac{Q}{Q}, & \lambda = 0 \end{cases} . \]

The approximation in Equation (6) can speed the calculation. However, the approximate analytic solutions are only effective when \( \lambda \) is around zero. When \( \lambda \) is not around zero, the investor still can get the prices of the fuzzy vulnerable call and put by using the integrated formula instead of the approximate solutions.

**Theorem 3:** \((\lambda, \lambda^*)\)-interval bounds of a fuzzy vulnerable option

The membership of a fuzzy subset is derived according to Definition 3. Under a non-additive \( \lambda \)-additive fuzzy measure, the \((\lambda, \lambda^*)\)-interval bounds of the vulnerable call and the vulnerable put option prices are derived. The lower bound of the expected value is computed using the lower Choquet integral, and the upper bound is obtained by computing the corresponding upper Choquet integral.

\[ \left[ v_{c_t}^{\lambda}, v_{c_t}^{\lambda^*} \right] = \left[ e^{-r(T-t)} C_{\text{av}} Q_l \left( A_t \right), e^{-r(T-t)} C_{\text{av}} Q_l \left( A_t \right) \right] \]  

and

\[ \left[ v_{p_t}^{\lambda}, v_{p_t}^{\lambda^*} \right] = \left[ e^{-r(T-t)} C_{\text{av}} Q_l \left( B_t \right), e^{-r(T-t)} C_{\text{av}} Q_l \left( B_t \right) \right] \]  

Firstly, Carr et al. (2001) used the linear programming approach to discuss pricing bounds (bid-ask spread) for vulnerable options. In other words, it is useful for a market-maker to know the minimum (maximum) acceptable price. If the market-maker sets their ask (bid) price to this reserve price, they can be assured that all liabilities issued can be hedged so that the residual is acceptable. Secondly, Hui et al. (2003, 2007) used the Lie algebraic approach to develop a two (three)-factor valuation model of European options incorporating a stochastic default barrier. The closed-form pricing formulae of vulnerable European options based on the model are used to investigate the impact of default risk on option values. Finally, rather than using existing methods, this paper uses fuzzy measure theory or the Choquet integral approach and obtains pricing bounds (bid-ask spread) for the vulnerable option. The major contribution of this study is providing a subjective measure used by individual investor \( \lambda \), which used the risk attitude of market participants to explain pricing bounds.

Another viewpoint is provided in this article to explain the \((\lambda, \lambda^*)\)-interval bounds of fuzzy vulnerable options, i.e. some investors, which is termed as “overconfident” investors, has more confidence than others to take active trading strategies in some investment opportunities. This over-optimistic trading behavior also leads to heterogeneous beliefs or differences in opinions. We propose more detailed explanation about this concept in Subsection 4.2.
3.3 Greek letters and hedging strategies

Financial institutions selling options or other derivatives on the OTC markets face the problem of risk management. This section first discusses properties of Greek letters, each of which measures a different dimension of option position risk, where the aim of the trade is to manage the Greeks to ensure acceptable risk. Next, this study deals with hedging strategies in European call options.

3.3.1 Greek letters

Using the fuzzy vulnerable options formula in integration form, as shown in Equation (5), this study applies partial differentiation to the formula with respect to several parameters, including stock price, volatility, risk-free rate, and time to maturity, to obtain their Greeks. For example, the delta of an option, \( \Delta \), is defined as the rate of change of the option price with respect to that of the underlying asset and the assets of the counterparty. The deltas of a fuzzy vulnerable call option are represented by \( \Delta S \) and \( \Delta V \).

Corresponding to any given \( \lambda \), the \((\lambda, \lambda^*)\)-interval of the fuzzy vulnerable call option is expressed as, Equation (7a). Therefore the \((\lambda, \lambda^*)\)-interval of the Delta of fuzzy vulnerable call option, \( \Delta S \) and \( \Delta V \), are as follows:

\[
\Delta_S^\lambda = \left[ \Delta_{S,1}, \Delta_{S,2}^* \right] = \left[ \min \left( \frac{\partial \nu_c}{\partial S_i}, \frac{\partial \nu_c}{\partial S_j} \right), \max \left( \frac{\partial \nu_c}{\partial S_i}, \frac{\partial \nu_c}{\partial S_j} \right) \right],
\]

\[
\Delta_V^\lambda = \left[ \Delta_{V,1}, \Delta_{V,2}^* \right] = \left[ \min \left( \frac{\partial \nu_c}{\partial V_i}, \frac{\partial \nu_c}{\partial V_j} \right), \max \left( \frac{\partial \nu_c}{\partial V_i}, \frac{\partial \nu_c}{\partial V_j} \right) \right],
\]

and all these intervals can be represented using a fuzzy number. For example, the delta hedge ratio of fuzzy vulnerable call option, \( (\Delta_S^\lambda) \), can be derived as follows.

\[
\Delta_S^\lambda = \frac{\partial \nu_c}{\partial S} =
\]

\[
e^{-r(T-t)} \frac{1}{\lambda} \int_{\kappa}^{\nu} \left[ \frac{\partial (1+\lambda)^{N_2(b_1,b_2,\rho_{SV})} \partial N_2 \left( b'_1, b'_2, \rho_{SV} \right) \partial b'_1}{\partial S} \right] dx
\]

\[
+ \frac{1-\beta}{\lambda} V \int_{\kappa}^{\nu} \left[ \frac{\partial (1+\lambda)^{N_2(d_1,d_2,\rho_{SV})} \partial N_2 \left( d'_1, d'_2, -\rho_{SV} \right) \partial d'_1}{\partial S} \right] dx
\]

\[
= \frac{e^{-r(T-t)} \ln(1+\lambda)}{2\pi \lambda \sqrt{\sigma^2_\xi (1-\rho_{SV}^2) (T-t) S}} \int_{\kappa}^{\nu} \left[ (1+\lambda)^{N_2(b_1,b_2,\rho_{SV})} \exp \left( -\frac{b_1^2 - 2\rho_{SV} b'_1 x_1 + x_1^2}{2 \sqrt{1-\rho_{SV}^2}} \right) \right] dx dx
\]

\[
+ \frac{(1-\beta)V \ln(1+\lambda)}{2\pi \lambda \sqrt{\sigma^2_\xi (1-\rho_{SV}^2) (T-t) S}} \int_{\kappa}^{\nu} \left[ (1+\lambda)^{N_2(b_1,b_2,\rho_{SV})} \exp \left( -\frac{d_1^2 + 2\rho_{SV} d'_1 x_1 + x_1^2}{2 \sqrt{1-\rho_{SV}^2}} \right) \right] dx dx.
\]
\[ \Delta \nu \] can be derived in the same manner. This study also calculated the Gamma ratio, Rho ratio and Theta ratio for the fuzzy vulnerable call options, but these calculations are omitted here.

3.5.2 Hedging strategies

As shown in section 3.2, the price of the vulnerable option which is valued under heterogeneity is a fuzzy set. The value of \( \lambda \) is fixed arbitrarily. A hedging strategy is an \( M_t \)-predictable process \( \{(\theta_{0t}, \theta_{1t}, \theta_{2t})\}_{t \geq 0} \) with values in \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \), where \( \theta_{0t} \) denotes the riskless bond \( (R_t) \), \( \theta_{1t} \) and \( \theta_{2t} \) represent the asset underlying asset \( (S_t) \) and the underlying assets of the counterparty \( (V_t) \) at time \( t \), respectively. Hence, the strategy satisfies

\[
\pi_{t, \lambda}^+ = \theta_{0t} R_t + \theta_{1t} S_t + \theta_{2t} V_t, \quad t \geq 0,
\]

where \( \pi_{t, \lambda}^+ = e^{-\lambda(t-t')} CE_{t'}^{t, \lambda} [A_t] \) is called a wealth process. A \( (\lambda, \lambda^*) \)-interval hedging strategy \( \{(\theta_{0t}, \theta_{1t}, \theta_{2t})\}_{t \geq 0} \) is self-financing if

\[
R_t d\theta_{0t}^+ + S_t d\theta_{1t}^+ + V_t d\theta_{2t}^+ = 0.
\]

Here, the \( M_t \)-predictable process \( \{(\theta_{0t}, \theta_{1t}, \theta_{2t})\}_{t \geq 0} \) satisfy some usual conditions:

\[
\int_0^T |\theta_{0t}^+| dt < \infty, \int_0^T (\theta_{1t}^+) dt < \infty, \text{ almost surely and } \int_0^T (\theta_{2t}^+) dt < \infty \text{ almost surely.}
\]

The intuitive meaning of (9) is that changes in the holdings of the bond \( R_t d\theta_{0t}^+ \) can only take place due to corresponding changes in the holdings of the asset underlying \( S_t d\theta_{1t}^+ \) and the underlying assets of the counterparty \( V_t d\theta_{2t}^+ \). Consequently, capital inflow and outflow do not exist. For a self-financing \( (\lambda, \lambda^*) \)-interval hedging strategy \( \{(\theta_{0t}, \theta_{1t}, \theta_{2t})\}_{t \geq 0} \), we have

\[
d\pi_{t, \lambda}^+ = \theta_{0t}^+ dR_t + \theta_{1t}^+ dS_t + \theta_{2t}^+ dV_t, \quad t \geq 0,
\]

and

\[
\pi_{t, \lambda}^+ = \theta_{0t}^+ R_t + \theta_{1t}^+ S_t + \theta_{2t}^+ V_t + \int_0^t \theta_{0t}^+ dR_s + \int_0^t \theta_{1t}^+ dS_s + \int_0^t \theta_{2t}^+ dV_s.
\]

Then, this study thus obtains the following results.

**Theorem 4:** The \( (\lambda, \lambda^*) \)-interval hedging strategies \( \{(\theta_{0t}, \theta_{1t}, \theta_{2t})\}_{t \geq 0} \) for the fuzzy vulnerable call option with measure \( \lambda \) are given by

\[
\{\theta_{0t}, \theta_{1t}, \theta_{2t}\}_{t \geq 0} = \{c_{t}^{\lambda^*(\lambda)} - \Delta_S V_t, \Delta_S V_t, \Delta_V V_t\}_{t \geq 0}
\]

**Proof:** The proof resembles Theorem 7.6.2 presented in Elliott and Kopp (1999) for the model of Black-Scholes (1973). For hedging purposes, since a perfect hedge is impossible, only a range of option price values can be found, and the best effort is to make the \( (\lambda, \lambda^*) \)-interval hedging strategies in Equation (10) on the option price range as wider as possible.
3.4 The economic meaning of using a $\lambda$-additive fuzzy measure

![Image](image_url)

Figure 1. This figure shows the relationship between the parameter $\lambda$ and the risk attitude of market participant. $\lambda < 0$ means that the attitude is risk-seeking. $\lambda = 0$ means that the attitude is risk-neutral. $\lambda > 0$ means that the attitude is risk-averse.

Notably, $\lambda = 0$, thus $\lambda^* = 0$, which means that the investors are risk-neutral and homogenous, and thus the price bounds degenerate to the standard Black-Scholes’ no arbitrage price in the complete market, as in Klein (1996). Moreover, $\lambda \neq 0$ means that the investors are not risk-neutral. Furthermore, Figure 1 shows the relationship between the parameter $\lambda$ and the risk attitude of the market participant. $\lambda < 0$ means that they are risk-seeking, and optimistic about the future of the market, while $\lambda > 0$ means that they are risk-averse and pessimistic. $\lambda$ thus represents risk attitude and heterogeneous expectations basically come from this.

3.5 Distinguishing credit and liquidity risk

Firstly, liquidity risk denotes financial risk arising from uncertain liquidity. An institution loses liquidity if its credit rating falls, it experiences sudden unexpected cash outflows, or some other event causes counterparties to avoid trading with or lending to it. A firm is also exposed to liquidity risk if markets on which it depends suffer loss of liquidity. Market participants use the bid-ask spread (or pricing bounds) to measure asset liquidity. Different products can be compared using the ratio of the spread to the mid price of the product. Reducing ratio is associated with increasing asset liquidity. Hence, the bid-ask spread in Equation (6a) (or (6b)) can be used to measure the fuzzy vulnerable call (or put) option with liquidity risk.

Second, credit risk denotes the risk investors’ face of suffering losses from failing to make payments as promised, and thus is a risk faced by all buyers of call options. Hence, credit risk offers a measure of the difference in changes in call option price between the fuzzy models of Black-Scholes (1973) and Klein (1996). Applying this fuzzy method to the standard Black-Scholes (1973) model with a final payoff of $C_T = \text{Max}(S_T - K, 0)$ yields the fuzzy call option pricing formula (Han and Zheng, 2005):

$$C^{\lambda \neq 0} = S_0 \mu \left[ N\left( d \right) \right] \left( 1 + \lambda \right)^{N(d-\sigma \sqrt{T-t}) - N(d)} - Ke^{-r(T-t)} \mu \left[ N(d - \sigma \sqrt{T-t}) \right]$$

(11)

where

$$d = \frac{\ln\left( \frac{S}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right)(T-t)}{\sigma \sqrt{T-t}}$$

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Hence, the measurement of credit risk can be defined as the differences between Equations. (6) and (11) identified when applying measure $\lambda$. Liquidity risk usually tends to compound other risks. If a trading organization has a position in an illiquid asset, its limited ability to liquidate that position at short notice compounds its market risk. Suppose a firm has offsetting cash flows with two different counterparties on a given day. If the counterparty owes a payment defaults, the firm must raise cash from elsewhere to make its payment. Failure to do this will see the firm default. In this case, liquidity risk compounds credit risk. Finally, this study distinguishes credit and liquidity risk, as shown in Figure 2. Table 1 lists the effects of $\lambda \ (\geq 0)$ on the fuzzy vulnerable call option price.

![Diagram](image)

**Figure 2. Distinguish between credit and liquidity risk**

Comparing the proposed model to that of Klein (1996) yields the effect of liquidity risk. Comparing the proposed model with that of Han and Zhang (2005), or comparing the model of Klein (1996) with that of Black and Scholes (1973) yields the effect of default risk. The relationship among the models can be expressed as shown in Figure 3.

**4. Numerical examples**

In Section 4.1, we first explain the advantages of using a $\lambda$-additive fuzzy measure. In Section 4.2, we then consider the sensitivity and numerical analyses of fuzzy vulnerable options. Finally, in Section 4.3, we give an application of using fuzzy vulnerable options to price employee stock options whose underlying asset is non-traded or only thinly traded.

![Diagram](image)

**Figure 3. Model comparison**
4.1 The advantages of a $\lambda$-additive fuzzy measure

Probability theory provides a quantitative tool for randomness while possibility theory provides a good qualitative tool for incompleteness. Probability and possibility capture different facets of uncertainty. As stated by Dubois et al. (1993), it is useful and interesting to transform a probability measure to a possibility measure, as the latter are simpler, and as a way to deal with information incompleteness. Due to a lack of information, investment problems are often uncertain or vague in a number of ways. This type of uncertainty has long been handled utilizing by probability theory or statistics. However, in many areas, such as funds, stocks, debt, derivatives and others, individuals’ judgment of events may be significantly different based on subjective perceptions or personality tendencies, and thus it is often fuzzy. Hence, using fuzzy approaches to describe and eliminate this “fuzziness” which is inherent in the subjective assessments made by investors in the option pricing model seems to be an appropriate method.

This ability to describe investors’ risk attitudes and subjective beliefs makes the fuzzy measure superior to other one-parameter approaches, such as the pricing measure coming from a constant relative risk aversion (CRRA) utility function with one parameter, $R$. The difference between our chosen measure and other methods is that our parameter, $\lambda$, represents the risk attitude of an investor and the bounds of option prices that the investor can refer to when making investment decisions. The selection of the value of $\lambda$ is not arbitrary, but instead should be chosen according to the investor’s personality and the prevailing market atmosphere.

4.2 Sensitivity analysis

Table 1 shows the effects of $\lambda$ on the fuzzy vulnerable call option price. The curve in Figure 4 illustrates the relationship between the price of fuzzy vulnerable call options, the deadweight cost of bankruptcy, $\beta$, and the degree of fuzziness, $\lambda$. We found that when $\lambda$ approaches zero, the upper and lower bounds of the $(\lambda, \lambda^*)$-interval are convergent to those of the Klein model in all cases. The $(\lambda, \lambda^*)$-interval degenerate to one single option value as that in the Klein model when $\lambda$ tends to zero. $\lambda$ can be thought of as a measure of the investor’s risk attitude, and zero-$\lambda$ denotes that the risk attitude of the investor is neutral. Moreover, when $\lambda$ is zero, there is no information ambiguity between market participants, and all of them have “identical rationality” and their own market prices are all the same (3.6901), when $\beta = 0.1$, the price of the fuzzy vulnerable call option is the same as in those in Klein’s model.
Figure 4. The effect of information ambiguity on the value of the Klein’s (1996) vulnerable option. Parameters $S = 35$, $V = 10$, $D = D^* = 10$, $K = 35$, $\sigma_S = \sigma_V = 0.30$, $r = 0.05$, $T - t = 0.75$, $\rho_{SV} = 0.5$

The upper bound and lower bound of the $(\lambda, \lambda^*)$-interval have their special meaning in fuzzy model construction. The upper bound plays a role as the long position of the buyer and the lower bound plays a role as the short position of the writer. There are three trader positions in the fuzzy model, i.e. buyer position, writer position and broker position. Daniel, Hirshleifer and Subrahmanyam (1998) used overconfidence to explain the predictable returns of financial assets, while Odean (1998) demonstrated that overconfidence can cause excessive trading. Tables 1 reveals that when the dead-weight loss rate, $\beta$, changes from 0.1 to 0.9, the $(\lambda, \lambda^*)$-interval is narrow. This occurs because the price of vulnerable options reduces with increasing dead-weight loss rate. The $(\lambda, \lambda^*)$-interval can also be viewed as the ask-bid spread, which can be a measure of liquidity risk.

For risk-averse brokers, the $(\lambda, \lambda^*)$-interval is larger, consistent with the financial concept that the ask-bid spread increases with the liquidity risk. In the model, the liquidity risk is subjective and judged by the broker, and thus option prices or option price intervals vary among different individuals, according to their thinking. Moreover, options that are subject to credit risk are more risky than risk-free options. That is, the bid-ask spread of the fuzzy vulnerable call option exceeds that of the fuzzy Black-Scholes call option (Han and Zheng, 2005), as shown in Table 1.

4.3 Using fuzzy vulnerable options to price employee stock options

We can apply the fuzzy vulnerable option model proposed in this article to price the fair value of employee stock options. In financial statements, a company may price its own employee stock options with the Black-Scholes model. However, this has some shortcomings, as since the stock option is issued by the employee’s company, the writer (i.e., the company) may default in the future. Thus, the stock option should be priced with the correlated credit risk. We can get the default probability from the market value balance sheet by assuming the asset value follows the geometric Brownian motion. In addition, the premium of the employee stock option also needs to be considered, and we can measure this with $\lambda$ in the fuzzy vulnerable option model, and thus we can obtain the appropriate price of such options. For example, if we simply ignore dilution effects and assume the stock price is $50, the exercise price is $50, the capital structure of the company is 50%-50%, the risk-free rate is 3%, the dead-weight loss rate is 0.25, the volatility of the stock is 30%, the volatility of the asset is 20%, the time to maturity is two years, the correlation between the stock and asset is 0.5, and the employee stock option is the European-type, then the Black-Scholes call price is $9.69, and we assume that the market price of the call option on the company’s stock is $12. We can
thus get that $\lambda$ is 0.74, the measure of the premium, from the fuzzy vulnerable option model, and can then use this and other parameters with our model to find that the fair value of an employee stock option is $10.71.

Table 1. Price of fuzzy vulnerable call options

<table>
<thead>
<tr>
<th>Parameter Settings</th>
<th>Fuzzy vulnerable call options</th>
<th>Fuzzy call options</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda = -0.5$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td>$\beta = 0.1$</td>
<td>2.6747 1.6827</td>
<td>1.4887 0.9428</td>
</tr>
<tr>
<td>$\beta = 0.9$</td>
<td>4.8997 3.0586</td>
<td>7.1133 4.4173</td>
</tr>
<tr>
<td>$S = 30$</td>
<td>1.0899 0.3004</td>
<td>0.5886 0.3690</td>
</tr>
<tr>
<td></td>
<td>2.0793 0.6844</td>
<td>3.1102 1.9393</td>
</tr>
<tr>
<td>$V = 5$</td>
<td>4.9955 3.1484</td>
<td>2.8812 1.8339</td>
</tr>
<tr>
<td>$V = 15$</td>
<td>1.1513 0.1783</td>
<td>0.9251 0.1082</td>
</tr>
<tr>
<td></td>
<td>2.5019 0.2988</td>
<td>3.3977 0.4097</td>
</tr>
<tr>
<td>$S = 40$</td>
<td>4.6631 2.9397</td>
<td>2.7041 1.7225</td>
</tr>
<tr>
<td></td>
<td>8.1606 5.0859</td>
<td>11.481 7.0992</td>
</tr>
<tr>
<td>$V = 7.5$</td>
<td>1.4215 0.8928</td>
<td>0.7669 0.4835</td>
</tr>
<tr>
<td></td>
<td>2.6990 1.6876</td>
<td>4.0227 2.5070</td>
</tr>
<tr>
<td>$\sigma_s = 0.15$</td>
<td>1.5945 1.0040</td>
<td>0.9032 0.5734</td>
</tr>
<tr>
<td></td>
<td>2.8623 1.7857</td>
<td>4.0971 2.5393</td>
</tr>
<tr>
<td>$\sigma_f = 0.45$</td>
<td>3.7552 2.3612</td>
<td>2.0696 1.3088</td>
</tr>
<tr>
<td>$T - t = 0.5$</td>
<td>2.1641 1.3427</td>
<td>1.2068 0.7530</td>
</tr>
<tr>
<td></td>
<td>3.9545 2.4379</td>
<td>5.7324 3.5179</td>
</tr>
<tr>
<td>$\rho_{SV} = -0.5$</td>
<td>2.3758 0.9286</td>
<td>1.3246 0.5095</td>
</tr>
<tr>
<td></td>
<td>4.3438 1.7291</td>
<td>6.3004 2.5410</td>
</tr>
</tbody>
</table>

Parameters: $S = 35$, $V = 10$, $D = D^* = 10$, $K = 35$, $\sigma_s = \sigma_f = 0.30$, $r = 0.05$, $T - t = 0.75$, $\rho_{SV} = 0.5$
5. Conclusions

The main contribution of this study is to discuss the problem of credit risk and fuzziness. Credit risk is increasingly important, since defaults are increasingly common in modern financial markets, it is better to price options subject to credit risk when the traded options are not really risk-free, such as those sold by banks or dealers. Non-identical rationality is normal in financial markets, and should be included in the option pricing model to obtain a more reasonable reference price. Since each investor has its own subjective belief about the future market condition, unique probability measure is no longer suitable to deal with this problem. Instead, the $\lambda$-additive measure and the Choquet integration have been employed to derive the option pricing model and incorporate the fuzziness into it. Arguably, the Black-Scholes model makes no assumptions regarding risk preference, and only assumes a complete market and thus that the option can be priced in a risk-neutral world. However, these assumptions ignore subjective prices of investors, who may buy or sell options when the price reaches what they perceive as fair value. Considering investor risk preferences offers one meaning of reflecting this subjectivity. This study provides a pricing formula for integrated fuzzy vulnerable Black-Scholes options, and this formula yields an approximate analytic solution is suitable when $\lambda$ is around zero, as is generally the case for investors. Some concepts of behavior finance, such as “non-identical rationality” and “over-confidence” of market investors, are also proposed in this article to explain liquidity risk when evaluating financial derivatives in the OTC markets. Additionally, this study also presents a sensitivity analysis of the hedge strategies of fuzzy vulnerable options. Moreover, this study presents some numerical examples to clarify the meaning of the pricing formula and applies the fuzzy vulnerable option model to value the fair price of an employee stock option.

This study also has some limitations. Notably, like other structural form credit risk models, the proposed model can only assess the default condition at the time of option maturity. To resolve this drawback, numerous researchers incorporate the jump process into the model to enable “instant default”. However, incorporating the jump process requires more complex mathematical techniques, the development of which is a matter for future research. To achieve the ability of “instant default”, the model can also be improved by fuzzifying the model parameters, since some degree of market uncertainty must be reflected in the model description. For example, financial market parameters (e.g. interest rates and volatility) fluctuate and expert opinions on the future movements of these parameters may differ. Using the theory of fuzzy numbers and stochastic analysis enables us to consider numerous sources of uncertainty, rather than only probabilistic uncertainty. The theory is also valuable because it provides a means of empirically quantifying the correctness of the proposed model by using market data and testing the heterogeneous belief models presented in this study. Once again, future researchers may want to investigate these issues.
References


Appendix

The derivation of a fuzzy vulnerable option is shown below. This article uses the Choquet integration to get the fuzzy vulnerable options formula in an integrated form and derives their approximate analytic solutions and this approximation can accelerate the calculation. The vulnerable call option price for Equation (6), can be rewritten as

\[
vc^{\lambda=0} = e^{-r(T-t)} \int_{-\infty}^{\infty} \left[ \left( 1 + \lambda \right) N_2(b'_1, b'_2, \rho_{SV}) - 1 \right] dx + \frac{(1-\beta)^{\lambda}}{D} \int_{-\infty}^{\infty} \left[ (1 + \lambda) N_2(d'_1, d'_2, -\rho_{SV}) - 1 \right] dx
\]

(A1)

Where

\[
b'_1 = \ln \left( \frac{S}{K} \right) + \frac{r - \frac{1}{2} \sigma_s^2}{\sigma_s \sqrt{T-t}} (T-t), \quad d'_1 = \ln \left( \frac{S}{K} \right) + \frac{r - \frac{1}{2} \sigma_s^2 + \rho_{SV}}{\sigma_s \sqrt{T-t}} (T-t),
\]

and the relationship of the \( \lambda \)-addictive fuzzy measure is

\[
\mu_{\lambda}(Q) = \begin{cases} 
Q, & \lambda = 0, \\
\frac{1}{2} \left[ (1 + \lambda)^0 - 1 \right], & \lambda \neq 0.
\end{cases}
\]

We use the integration technique known as integration by parts for Equation (A1) and expand it in a Taylor’s series around \( \lambda = 0 \), and then the approximate analytic solution is

\[
vc^{\lambda=0} = S_t \mu_{\lambda} \left[ N_2(a_1, a_2, \rho) \right] \left( 1 + \lambda \right) N_2(b_1, b_2, \rho) - Ke^{-r(T-t)} \mu_{\lambda} \left[ N_2(b_1, b_2, \rho) \right]
\]

\[
\left. + \frac{(1-\beta)^{\lambda}}{D} \left[ S_t e^{r(T-t) \rho_{SV} \sigma_s \sigma_Y} \mu_{\lambda} \left[ N_2(c_1, c_2, -\rho_{SV}) \right] \left( 1 + \lambda \right) N_2(d_1, d_2, -\rho_{SV}) - N_2(c_1, c_2, -\rho_{SV}) \right] \right]
\]

(A2)