Portfolio Selection Problems Using the Scenario Model with Fuzzy Returns

Takashi Hasuike*, Hiroaki Ishii

Graduate School of Information Science and Technology, Osaka University, Japan

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Abstract

In this paper, we propose several mathematical models with respect to portfolio selection problems, particularly using the scenario model including the ambiguous factors. These mathematical programming problems with probabilities and possibilities are called to stochastic programming problem and fuzzy programming problem, and it is difficult to find the global optimal solution for those problems. We manage to develop an efficient solution method to find the global optimal solution of such a nonlinear programming problem. Furthermore, a numerical example of the portfolio selection problem is given to compare our proposal models with previous standard fuzzy portfolio models.

Keywords: Portfolio selection problem, scenario model, stochastic and Fuzzy programming, nonlinear programming problem

1. Introduction

In recent investments, not only big companies, institutional investors but also individual investors called Day-Traders invest in stock, currency, land and property. Therefore, the role of investment theory called portfolio theory becomes more and more important. The portfolio selection problem considers selecting a combination of securities among portfolios containing a large number of securities satisfying the goal of obtaining the total return. So far, various studies for portfolio selection problems have been done. As for the research history on mathematical approach, Markowitz (1959) has proposed the mean-variance analysis model. It has been central to research activity in the real financial field and numerous researchers have contributed to the development of modern portfolio theory (cf. Luenberger (1997), Campbell et al. (1997), Elton and Gruber (1995). On the other hand, many researchers have proposed models of portfolio selection problems which extended Markowitz model; Capital Asset Pricing Model (CAPM) (Sharpe (1964), Lintner (1965), Mossin (1966)), mean-absolute-deviation model (Konno (1990), Konno, et al. (1993)), semi-variance model (Bawa and Lindenberg, 1977), safety-first model (Elton and Gruber, 1995), Value at Risk and conditional Value at Risk model (Rockafellar, 2000), etc.

Particularly, mean-variance model is formulated as a quadratic programming problem minimizing the total variance, and the use of large-scale mean-variance models are restricted to the stock portfolio selection in spite of the recent development in computational and modeling technologies in the financial engineering. Under such a situation, a mean-absolute deviation model has been proposed. This model is formulated as a linear programming problem and can be solved faster than a corresponding mean-variance model. Furthermore, by

* Corresponding author. E-mail: thasuike@ist.osaka-u.ac.jp
the compact factorization of covariance matrices using the historical dates or the scenarios, the mean-absolute deviation model has been solved more efficiently.

On the other hand, in previous studies using scenario models, each return in scenarios are considered as fixed values. However, in the case where investors predict each of the future returns and decide an optimal portfolio, most of them often consider not only statistical analysis based on historical data but also their subjective intuitions derived from long-term experience in practical investments. Furthermore, considering the subjectivity and their psychological aspects, their prediction of each return includes some ambiguity such as “This return approximately increases up to 10 percent. However, it is possible that it increases up to 15 percent or 5 percent.” Therefore, we need to consider that each future return is set as not a fixed value but an ambiguous number. In this paper, we deal with the ambiguous numbers as fuzzy numbers, and proposed the portfolio selection problem by including them.

In mathematical programming problems, these models with probabilities and possibilities are called stochastic programming problems and fuzzy programming problems, respectively. There are some basic researches considering those problems with respect to portfolio selection (Bilbao-Terol et al., 2006), Carlsson (2002), Guo (1998), Huang (2006), Inuiuguchi and Lamik (2000), Inuiuguchi and Tanino (2000), Katagiri et al. (2004; 2005), Tanaka (2000), Watada (1997)). In previous researches, they are considered that expected return and its variance are assumed to be fuzzy variables. However, there are few models considering multi-scenario model involving fuzzy numbers directly and solving it analytically. Therefore, we propose fuzzy mean-variance model and mean-absolute deviation model using multi-scenario by including fuzzy numbers and construct the analytical solution method.

This paper is organized as follows. In Section 2, we introduce two portfolio selection problems using the scenario, mean-variance model and mean-absolute deviation model. In Sections 3 and 4, we extend these models of the portfolio selection problem by introducing fuzzy numbers. Furthermore, in order to compare our model with other previous basic and fuzzy portfolio models, we give a numerical example in Section 5. Finally in Section 6, we conclude this paper and discuss future studies.

2. Portfolio selection problem using the scenarios of returns

First, we introduce one of the traditional mathematical approaches for portfolio selection problems, a mean-variance model called Markowitz model. Markowitz has proposed the following mathematical programming problem as a portfolio selection problem:

\[
\begin{align*}
\text{Minimize} & \quad V(x) = x' \Sigma x \\
\text{subject to} & \quad E(x) = \bar{\mu}' x \geq r_G \\
& \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad x \geq 0
\end{align*}
\]

where

\[
\begin{align*}
\bar{\mu} & : \text{Mean value of } n\text{-dimensional Gaussian random variable row vectors} \\
\Sigma & : \text{Gaussian random variance} \\
r_G & : \text{Minimum value of the goal for expected total return} \\
a_j & : \text{Cost coefficient of asset } j \\
b & : \text{Maximum value of total budget} \\
x & : \text{Purchasing volume (an } n\text{-dimensional decision variable column vector)}
\end{align*}
\]
This formulation has long served as the basis of financial theory. This problem is a quadratic programming problem, and so we find the optimal portfolio using standard nonlinear programming approaches. However, it is not efficient to solve the large scale quadratic programming problem directly. Furthermore, in the case where an investor expects the future return of each product, she or he does not consider only one scenario of the future return, but often several scenarios.

In this regard, using many scenarios of the future returns, the mean-variance model for the portfolio selection problem has been reformulated. Let \( r_j \) be the realization of random variable \( R \) about the scenario \( i \), \( (i = 1, 2, \ldots, m) \), which we assume to be available from historical data and from subjective prediction of decision makers. Then, the return vector of scenario \( i \) is as follows:

\[
\mathbf{r}_i = (r_{i1}, r_{i2}, \ldots, r_{in}), \quad i = 1, 2, \ldots, m
\]

where \( n \) is the number of total asset. We introduce the probabilities for each scenario as follows:

\[
p_i, \quad i = 1, 2, \ldots, m
\]

We also assume that the expected value of the random variable can be approximated by the average derived from these data. Particularly, we let

\[
\mathbf{r}_j E[R_j] = \sum_{i=1}^{m} p_i r_{ij}
\]

Then, the mean value \( E(x) \) and variance \( V(x) \) derived from the data are as follows:

\[
E(x) = \sum_{j=1}^{m} \bar{r}_j x_j = \sum_{j=1}^{m} \left( \sum_{i=1}^{m} p_i r_{ij} \right) x_j
\]

\[
V(x) = \sum_{i=1}^{m} p_i \left( \sum_{j=1}^{n} r_{ij} x_j - E(x) \right)^2 = \sum_{i=1}^{m} p_i \left( \sum_{j=1}^{n} r_{ij} x_j - \sum_{i=1}^{m} \left( \sum_{j=1}^{n} p_i r_{ij} \right) x_j \right)^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jk} x_j x_k
\]

For simplify of the following discussion, we assume each probability \( p_i \) to have the same value of \( \frac{1}{m} \). From the above parameters, we transformed the Markowitz model into the following problem:

\[
\text{Minimize} \quad \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \left( r_{ij} - \bar{r}_j \right) x_j \right)^2
\]

subject to

\[
\bar{r}_j = \frac{1}{m} \sum_{i=1}^{m} r_{ij}
\]

\[
\sum_{j=1}^{n} \bar{r}_j x_j \geq r_G, \sum_{j=1}^{n} a_j x_j \leq b, \quad x \geq 0
\]
Furthermore, introducing parameters \( z_i = \sum_{j=1}^{n} (r_{ij} - \bar{r}_j) x_j \), \((i = 1, 2, ..., m)\), we equivalently transformed problem (7) into the following problem:

Minimize \( \frac{1}{m} \sum_{i=1}^{m} z_i^2 \)

subject to \( z_i - \sum_{j=1}^{n} (r_{ij} - \bar{r}_j) x_j = 0, (i = 1, 2, ..., m) \)
\( \bar{r}_j = \frac{1}{m} \sum_{i=1}^{m} r_{ij}, \)
\( \sum_{j=1}^{n} \bar{r}_j x_j \geq r_G, \sum_{j=1}^{n} a_j x_j \leq b, x \geq 0 \) \( (8) \)

Since this problem is a quadratic programming problem not to include the variances, we set each parameter and solve it more efficiently than the Markowitz model. However, this problem is not an easy linear programming problem.

Konno (1990) has proposed the Mean-absolute deviation model for portfolio selection problems. This problem is formulated as a linear programming problem and it is essentially equivalent to the traditional portfolio model if the rate of the return of assets is multivariate normally distributed. Now, let the absolute deviation (AD)

\[ AD = \frac{1}{m} \sum_{i=1}^{m} \left| \sum_{j=1}^{n} (r_{ij} - \bar{r}_j) x_j \right| = \frac{1}{m} \sum_{i=1}^{m} |r_i - \bar{r}_i| \] \( (9) \)

By introducing this definition of absolute deviation, he proposed the following portfolio selection problem:

Minimize \( \frac{1}{m} \sum_{i=1}^{m} \left| \sum_{j=1}^{n} (r_{ij} - \bar{r}_j) x_j \right| \)

subject to \( \bar{r}_j = \frac{1}{m} \sum_{i=1}^{m} r_{ij}, \)
\( \sum_{j=1}^{n} \bar{r}_j x_j \geq r_G, \sum_{j=1}^{n} a_j x_j \leq b, x \geq 0 \) \( (10) \)

Then, introducing parameters \( z_i = \sum_{j=1}^{n} (r_{ij} - \bar{r}_j) x_j \), problem (10) is transformed into the following problem based on the result of the previous study by Konno (1990):
Minimize \[ \frac{2}{m} \sum_{i=1}^{m} z_i \]

subject to \[ z_i + \sum_{j=1}^{m} r_j x_j \geq \sum_{j=1}^{m} \bar{r}_j x_j, (i = 1, 2, \ldots, m) \] \[ \bar{r}_j = \frac{1}{m} \sum_{j=1}^{m} r_j, \] \[ \sum_{j=1}^{m} \bar{r}_j x_j \geq r_G, \sum_{j=1}^{m} a_j x_j \leq b, x \geq 0 \] (11)

Consequently, by using this mean-Absolute deviation model, we easily solve a large scale portfolio selection problem.

3. Fuzzy extension of Mean-Variance portfolio selection problem

In previous studies, each return of scenarios is considered a fixed value derived from a random variable. However, considering the psychological aspects of decision makers, it is difficult to predict the future return as the fixed value. Therefore, we need to consider that the future return is ambiguous. Then we propose the portfolio selection problem using the scenarios where the return is ambiguous. In this paper, the return including the ambiguity is assumed to be a triangular fuzzy number as

\[ \tilde{r}_{ij} = \langle \bar{r}_{ij}, \alpha_j, \alpha_j \rangle = \langle r_{ij}, \alpha_j \rangle \] (12)

In this paper, for simplify of the following discussion, we assume \( r_{ij} - (1-h)\alpha_j \) to be a positive value. In the basic mean variance problem (8), since \( r_{ij} \) is a fuzzy variable, the objective function and the parameters including the fuzzy variable \( r_{ij} \) are also assumed to be fuzzy variables. Therefore, we cannot optimize this problem without transforming the objective function into another form.

In previous studies, some criteria with respect to fuzzy portfolio selection problems have been proposed. For example, Liu (2002; 2004) and Huang (2006) have proposed a portfolio selection problems using fuzzy or hybrid (fuzzy and random) expected value and its variance. Katagiri et al. (2005) have proposed a portfolio selection problem using possibility measure...
and probability measure. Carlsson (2002) has proposed a portfolio selection problem using the possibility mean value. In this paper, we assume the following cases:

(a) Since the main objective is to minimize the total variance and the investor considers that she or he manages to minimize it as small as possible even if the aspiration level becomes smaller.

(b) On the other hand, it is clear that the investor also considers that she or he never fails to earn the total return more than the goal in the variance constraint.

Therefore, we consider a fuzzy portfolio selection problem for probabilistic expected value and variance. Then, we introduce a possibility measure for the total variance based on assumption (a) and a necessity measure for the expected total return based on assumption (b).

Then, we convert the basic mean variance problem (7) into the following problem including the chance-constraints:

\[
\begin{align*}
\text{Minimize} & \quad \sigma_G \\
\text{subject to} & \quad \text{Pos}\{\tilde{V}(x) \leq \sigma_G\} \geq h, \\
& \quad \text{Nec}\{\tilde{E}(x) \geq r_G\} \geq h, \\
& \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad x \geq 0
\end{align*}
\]

where \(\text{Pos}\{\tilde{V}(x) \leq \sigma_G\}\) is a possibility measure and this means \(\text{Pos}\{\tilde{V}(x) \leq \sigma_G\} = \sup_{\sigma \in \tilde{V}(x)} \mu_{\tilde{V}(x)}(\sigma)\), then \(\text{Nec}\{\tilde{E}(x) \geq r_G\}\) is a necessity measure and this means \(\text{Nec}\{\tilde{E}(x) \geq r_G\} = 1 - \sup_{r \geq r_G} \mu_{\tilde{E}(x)}(r)\). In this problem, membership functions \(\mu_{\tilde{E}(x)}(r)\) and \(\mu_{\tilde{V}(x)}(\sigma)\) are assumed to be the following forms using the fuzzy extension principle. First, fuzzy number \(\tilde{E}(x)\) is given as the following triangular fuzzy numbers:

\[
\tilde{E}(x) = \left\{ \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \alpha_i r_i \right) x_j, \sum_{j=1}^{n} \alpha_j x_j \right\}
\]

Therefore, we obtain membership function \(\mu_{\tilde{E}(x)}(r)\) as follows:

\[
\mu_{\tilde{E}(x)}(\omega) =
\begin{cases}
\frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \alpha_i r_i \right) x_j - \omega & \text{if } q \leq \omega \leq q^\prime \\
\frac{\omega - \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_i x_j}{\sum_{j=1}^{n} \alpha_j x_j} & \text{if } q \leq \omega \leq q^\prime \\
0 & \text{if } \omega < q^-, q^+ < \omega
\end{cases}
\]
Next, we consider membership function $\mu_{\tilde{r}_i}(\sigma)$. In a way similar to $\mu_{E(x)}(r)$, we obtain the fuzzy number for $\bar{z}_i = \sum_{j=1}^n (\tilde{r}_j - \bar{r}_j)x_j = \sum_{j=1}^n \tilde{r}_j x_j - \tilde{E}(x)$ as the following triangular fuzzy number:

$$\bar{z}_i = \left\langle \frac{1}{m} \sum_{j=1}^m \tilde{r}_j, 2 \sum_{j=1}^n \alpha_j x_j \right\rangle = \left\langle z_i, 2 \sum_{j=1}^n \alpha_j x_j \right\rangle$$

(16)

Therefore, we obtain membership function $\mu_{z_i}(\omega)$ as follows:

$$\mu_{z_i}(\omega) = \left\{ \begin{array}{ll}
\frac{z_i - \omega}{2 \sum_{j=1}^n \alpha_j x_j} & \left( z_i - 2 \sum_{j=1}^n \alpha_j x_j \leq \omega \leq z_i \right) \\
\frac{\omega - z_i}{2 \sum_{j=1}^n \alpha_j x_j} & \left( z_i \leq \omega \leq z_i + 2 \sum_{j=1}^n \alpha_j x_j \right) \\
0 & \left( \omega < z_i - 2 \sum_{j=1}^n \alpha_j x_j, z_i + 2 \sum_{j=1}^n \alpha_j x_j < \omega \right) \end{array} \right.$$  

(17)

Furthermore, we consider the membership function for $\tilde{S}_i = \left( \sum_{j=1}^n (r_j - \bar{r}_j)x_j \right)^2$. In general cases using fuzzy product, membership functions often become much complicate functions. Therefore, with respect to $\mu_{z_i}(\omega)$, we introduce the $h$ -cut of this membership function in order to represent membership functions briefly:

$$\bar{z}_i = \left[ \mu_{z_i}^{(L)}(h), \mu_{z_i}^{(R)}(h) \right] = \left[ \mu_{r^x_x}^{(L)}(h) - \mu_{E(x)}^{(R)}(h), \mu_{r^x_x}^{(R)}(h) - \mu_{E(x)}^{(L)}(h) \right]$$

(18)

where

$$\bar{r}_i = \left[ \mu_{r^x_x}^{(L)}(h), \mu_{r^x_x}^{(R)}(h) \right]$$

$$\tilde{E}(x) = \left[ \mu_{E(x)}^{(L)}(h), \mu_{E(x)}^{(R)}(h) \right]$$

From this $h$ -cut of this membership function, that of membership function $\mu_{\tilde{S}_i}(\omega)$ is the following form:
\[ \tilde{S}_i = \begin{bmatrix} \mu_{S_i}^{(L)}(h), \mu_{S_i}^{(R)}(h) \end{bmatrix} \]

\[ \begin{align*}
\mu_{S_i}^{(L)}(h) &= \min \left\{ 0, (\mu_{Z_i}^{(L)}(h))^2, (\mu_{Z_i}^{(R)}(h))^2 \right\} \\
\mu_{S_i}^{(R)}(h) &= \max \left\{ (\mu_{Z_i}^{(L)}(h))^2, (\mu_{Z_i}^{(R)}(h))^2 \right\}
\end{align*} \] (19)

Therefore, the membership function \( \mu_{\tilde{y}(x)}(\sigma) \) is given as

\[ \tilde{V}(x) = \left[ \frac{1}{m} \sum_{i=1}^{m} \mu_{S_i}^{(L)}(h), \frac{1}{m} \sum_{i=1}^{m} \mu_{S_i}^{(R)}(h) \right] \] (20)

By using these membership functions, we transform the problem (13) into the following problem:

Minimize \( \sigma \)

subject to

\[ \frac{1}{m} \sum_{i=1}^{m} \mu_{S_i}^{(L)}(h) \leq \sigma, \]

\[ \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} h_i \right) x_j - h \sum_{j=1}^{n} \alpha_j x_j \geq r, \]

\[ \sum_{j=1}^{n} a_j x_j \leq b, x \geq 0 \] (21)

where the left part of membership function \( \mu_{S_i}^{(L)}(h) \) is as follows:

\[ \begin{align*}
\mu_{S_i}^{(L)}(h) &= \left( \sum_{j=1}^{n} h_i x_j \pm (1-h) \sum_{j=1}^{n} \alpha_j x_j \right)^2 + \left( \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} h_i \right) x_j \mp (1-h) \sum_{j=1}^{n} \alpha_j x_j \right)^2 \\
&\quad - 2 \left( \sum_{j=1}^{n} h_i x_j \pm (1-h) \sum_{j=1}^{n} \alpha_j x_j \right) \left( \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} h_i \right) x_j \mp (1-h) \sum_{j=1}^{n} \alpha_j x_j \right) \\
&= \left( \sum_{j=1}^{n} h_i x_j - \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} h_i \right) x_j \right)^2 - 4(1-h) \left( \sum_{j=1}^{n} \alpha_j x_j \right) \left( \sum_{j=1}^{n} h_i x_j \right) \right) \] (22)

Then, introducing parameters \( z_i^{(h)} \), problem (21) is equivalently transformed into the following problem:

Minimize \( \frac{1}{m} \left( z_i^2 - 4(1-h) \left( \sum_{j=1}^{n} \alpha_j x_j \right) \left( \sum_{j=1}^{n} h_i x_j \right) \right) \)

subject to

\[ z_i - \sum_{j=1}^{n} h_i x_j + \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} h_i \right) x_j = 0, (i = 1, 2, ..., m) \] (23)

\[ \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} h_i \right) x_j - h \sum_{j=1}^{n} \alpha_j x_j \geq r, \]

\[ \sum_{j=1}^{n} a_j x_j \leq b, x \geq 0 \]
In problem (23), the objective function is a convex quadratic function, and so problem (23) is equivalent to a convex quadratic programming problem. Therefore, we obtain a global optimal solution by using standard convex programming approaches. Furthermore, in the case that each return does not include fuzziness, i.e., each $\alpha_j = 0, (j = 1, 2, ..., n)$, problem (23) is degenerated to the following problem:

\[
\text{Minimize } \frac{1}{m} z^2_j \\
\text{subject to } z_j - \sum_{j=1}^{n} \tilde{r}_{ij} x_j + \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \tilde{r}_{ij} \right) x_j = 0, (i = 1, 2, ..., m) \\
\frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} \tilde{r}_{ij} \right) x_j \geq r_G, \\
\sum_{j=1}^{n} a_j x_j \leq b, x \geq 0
\]

(24)

This problem is equivalent to the basic mean-variance portfolio selection problem (8). Consequently, we find that problem (23) is a fuzzy extended model for basic mean-variance model.

4. Fuzzy extension of mean-absolute deviation portfolio selection problem

In a way similar to Section 3, we consider the fuzzy extension of mean-absolute deviation portfolio selection problem. First, we rewrite the mean-absolute deviation model as

\[
\text{Minimize } \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} (r_{ij} - \tilde{r}_j) x_j \\
\text{subject to } \tilde{r}_j = \frac{1}{m} \sum_{i=1}^{m} r_{ij} \\
\sum_{j=1}^{n} \tilde{r}_j x_j \geq r_G, \sum_{j=1}^{n} a_j x_j \leq b, x \geq 0
\]

(25)

In this problem, the objective function and parameters including fuzzy variables $\tilde{r}_{ij}$ are also assumed to be fuzzy variables. Therefore, we convert problem (25) into the following problem by using chance constraints:

\[
\text{Minimize } \sigma_G \\
\text{subject to } \text{Pos} \{D(x) \leq \sigma_G \} \geq h, \quad \text{Nec} \{E(x) \geq r_G \} \geq h, \\
\sum_{j=1}^{n} a_j x_j \leq b, x \geq 0
\]

(26)
The same kind of membership functions is given in Section 3, so we consider membership function \( \mu_{\hat{\beta}(x)}(h) \). The \( h \)-cut of membership function becomes the following form in a way similar to Section 3:

\[
\hat{\beta}(x) = \left[ \mu_{\hat{\beta}(x)}^{(1)}(h), \mu_{\hat{\beta}(x)}^{(2)}(h) \right] = \left[ \frac{1}{m} \sum_{i=1}^{m} \mu_{\hat{\beta}(x)}^{(1)}(h), \frac{1}{m} \sum_{i=1}^{m} \mu_{\hat{\beta}(x)}^{(2)}(h) \right]
\]  

(27)

where

\[
\mu_{\hat{\beta}(x)}^{(1)}(h) = \begin{cases} 
\mu_{\hat{\beta}(x)}^{(1)}(h) & \left( \mu_{\hat{\beta}(x)}^{(1)}(h) \geq 0 \right) \\
0 & \left( \mu_{\hat{\beta}(x)}^{(1)}(h) < 0 \leq \mu_{\hat{\beta}(x)}^{(2)}(h) \right) \\
-\mu_{\hat{\beta}(x)}^{(2)}(h) & \left( \mu_{\hat{\beta}(x)}^{(2)}(h) < 0 \right)
\end{cases}
\]

and

\[
\mu_{\hat{\beta}(x)}^{(2)}(h) = \max \left\{ 0, -\mu_{\hat{\beta}(x)}^{(1)}(h), \mu_{\hat{\beta}(x)}^{(2)}(h) \right\} \left( \mu_{\hat{\beta}(x)}^{(1)}(h) < 0 \leq \mu_{\hat{\beta}(x)}^{(2)}(h) \right) \\
-\mu_{\hat{\beta}(x)}^{(2)}(h) \left( \mu_{\hat{\beta}(x)}^{(2)}(h) < 0 \right)
\]

and

\[
\begin{array}{l}
z_i = \left[ \mu_{\hat{\beta}(x)}^{(1)}(h), \mu_{\hat{\beta}(x)}^{(2)}(h) \right] \\
= \left[ \mu_{\hat{\beta}(x)}^{(1)}(h) - \mu_{\hat{\beta}(x)}^{(2)}(h), \mu_{\hat{\beta}(x)}^{(2)}(h) - \mu_{\hat{\beta}(x)}^{(1)}(h) \right] \\
= \left[ z_i - 2(1-h)\sum_{j=1}^{m} \alpha_j x_j, z_i + 2(1-h)\sum_{j=1}^{m} \alpha_j x_j \right]
\end{array}
\]

Using this membership function, problem (26) is transformed into the following problem:

Minimize \( \sigma_G \)

subject to \( \frac{1}{m} \sum_{i=1}^{m} \mu_{\hat{\beta}(x)}^{(1)}(h) \leq \sigma_G \)

\[
\frac{1}{m} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \bar{t}_i \right) x_j - h \sum_{j=1}^{n} \alpha_j x_j \geq r_G, \\
\sum_{j=1}^{n} \alpha_j x_j \leq h, x \geq 0
\]  

(28)

In order to solve problem (26) analytically, we introduce the following parameters:

\[
z_i = v_i - v_i^- + v_i^+, \\
-\bar{\alpha}(h) \leq v_i \leq \bar{\alpha}(h), \\
v_i^- = \bar{\alpha}(h) - z_i, \left( v_i^+ = 0, z_i \geq \alpha(h) \right), \\
v_i^+ = z_i - \left( -\bar{\alpha}(h) \right), \left( v_i^- = 0, z_i \leq -\alpha(h) \right), \\
\bar{\alpha}(h) = 2(1-h)\left( \sum_{j=1}^{n} \alpha_j x_j \right)
\]
By using these parameters, problem (28) is equivalently transformed into the following problem based on the study of King (1993):

Minimize \( \frac{1}{m} \sum_{i=1}^{m} (2z_i - 2\alpha(h)) \)

subject to

\[
\begin{align*}
    z_i &= v_i - v_i^+ + v_i^- \leq -\alpha(h) \leq \alpha(h), \\
    z_i - \alpha(h) &\geq 0, \\
    z_i + \alpha(h) &\geq 0, \\
    \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} F_{ij} x_j \right) - h \sum_{j=1}^{n} \alpha_j x_j &\geq r_G, \\
    \sum_{j=1}^{n} a_j x_j &\leq b, x \geq 0
\end{align*}
\]  

(29)

Since this problem is a linear programming problem, we obtain the global optimal solution by using linear programming approaches such as the Simplex method and Interior point method. Furthermore, in a way similar to the proposed fuzzy mean-variance model in Section 3, in the case where each return does not include fuzziness, i.e., each \( \alpha_j = 0, (j = 1, 2, \ldots, n) \), problem (29) is degenerated to the following problem:

Minimize \( \frac{2}{m} \sum_{i=1}^{m} (2z_i - 2\alpha(h)) \)

subject to

\[
\begin{align*}
    z_i - \left( \sum_{j=1}^{n} F_{ij} x_j - \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} F_{ij} \right) x_j \right) &\geq 0, \\
    \left( \sum_{j=1}^{n} F_{ij} x_j - \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} F_{ij} \right) x_j \right) - z_i &\geq 0, \\
    \frac{1}{m} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} F_{ij} \right) x_j &\geq r_G, \\
    \sum_{j=1}^{n} a_j x_j &\leq b, x \geq 0
\end{align*}
\]  

(30)

This problem is equivalent to the basic mean-absolute deviation model (11). Consequently, we find that problem (29) is a fuzzy extended model for the basic mean-absolute deviation model.

5. Numerical example

In order to compare our proposal model (13) with one of the most important portfolio models, the mean-variance model, Huang (2006) model using a credibility measure and Vercher (2007) model using a possibility mean value, let us consider an example shown in Table 1 based on data introduced by Markowitz (1959). We assume that there are nine assets whose returns are assumed to be symmetric triangle fuzzy numbers involving spread \( \alpha \). Then, we assumed that the number of scenarios with respect to each return is 10 and the spread \( \alpha \) for each return is equal among all scenarios.
Table 1. Sample data from Markowitz’s historical data and their fuzzy numbers.

<table>
<thead>
<tr>
<th>Returns</th>
<th>Center (mean) value</th>
<th>Spread $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0.066</td>
<td>0.03</td>
</tr>
<tr>
<td>R2</td>
<td>0.062</td>
<td>0.01</td>
</tr>
<tr>
<td>R3</td>
<td>0.146</td>
<td>0.06</td>
</tr>
<tr>
<td>R4</td>
<td>0.173</td>
<td>0.08</td>
</tr>
<tr>
<td>R5</td>
<td>0.198</td>
<td>0.1</td>
</tr>
<tr>
<td>R6</td>
<td>0.055</td>
<td>0.03</td>
</tr>
<tr>
<td>R7</td>
<td>0.128</td>
<td>0.02</td>
</tr>
<tr>
<td>R8</td>
<td>0.118</td>
<td>0.06</td>
</tr>
<tr>
<td>R9</td>
<td>0.116</td>
<td>0.06</td>
</tr>
</tbody>
</table>

In this numerical example, we introduce a maximum allocation rate as 0.2 with respect to each asset. We solve these problems and obtain optimal portfolios shown in Table 2.

Table 2. Optimal portfolio for each model.

<table>
<thead>
<tr>
<th></th>
<th>Proposed model</th>
<th>Basic model</th>
<th>Huang model</th>
<th>Vercher model</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0.200</td>
<td>0.075</td>
<td>0.200</td>
<td>0.145</td>
</tr>
<tr>
<td>R2</td>
<td>0.200</td>
<td>0.120</td>
<td>0.200</td>
<td>0.200</td>
</tr>
<tr>
<td>R3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>R4</td>
<td>0</td>
<td>0.200</td>
<td>0</td>
<td>0.200</td>
</tr>
<tr>
<td>R5</td>
<td>0.133</td>
<td>0.200</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>R6</td>
<td>0.067</td>
<td>0</td>
<td>0.200</td>
<td>0</td>
</tr>
<tr>
<td>R7</td>
<td>0.200</td>
<td>0.200</td>
<td>0</td>
<td>0.200</td>
</tr>
<tr>
<td>R8</td>
<td>0</td>
<td>0.195</td>
<td>0.200</td>
<td>0.200</td>
</tr>
<tr>
<td>R9</td>
<td>0.200</td>
<td>0.010</td>
<td>0.200</td>
<td>0.055</td>
</tr>
</tbody>
</table>

Table 2 shows that the optimal portfolio of our proposal fuzzy mean-variance model is similar to the basic mean-variance model and Huang model. In other words, we find that our model is an intermediate model between the basic model and Huang model. Furthermore, the optimal portfolio of our model tends to be selected more widely from small mean value assets to large mean value assets in the same way as the Vercher model.

6. Conclusion

In this paper, we have proposed some models of portfolio selection problems including fuzziness using the multi-scenario model. First, we have considered the fuzzy extension of mean-variance model for portfolio selection problems and shown that this problem has been transformed into a quadratic programming problem and included some basic portfolio selection problems. Furthermore, in a way similar to the fuzzy mean-variance model, we have proposed a fuzzy extended mean-absolute deviation model. We may be able to apply these proposed models to the case including not only fuzziness but also randomness, called fuzzy random variables.

As future studies, we need to consider not only the mean-variance model and mean-absolute deviation model but also other portfolio selection problems such as downside risk models and Value at Risk. Furthermore, we are now attacking the cases where the optimal solution is restricted to integers, and where we need to consider the portfolio selection problem is not only single but also multi-period.
References


