Unemployment, welfare and optimal monetary policy: convergence probability

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Using a rigorous convergence probability, this note shows why mathematic models are inconsistent with empirical evidence. Using a dynamic quadratic cointegration, a new theory of the concave utility shows that below the optimum, the volatility has a positive impact upon output. Beyond the optimum, the volatility of policy has a negative impact upon output and a positive impact upon the unemployment rate. In this new methodology, our equilibrium targets minimize inflation bias and fiscal bias. First, we compute the equilibrium and optimal policy; then reduce the heteroscedasticity, and finally estimate the convergence probability. Our surprising solution is individually rational and feasible while the budget is balanced over business cycles. For the problem of n>1 governments or agents, an agent estimates and proposes the maximum non-zero equilibrium which is most acceptable to all participants and outperforms the Nash noncooperative equilibrium. This solves the games and scientific problems unrelated to games as $n \to \infty$, regardless of whether the proposal is accepted voluntarily or not.

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1. Introduction

1.1 Motivations

The purpose of this paper is to investigate the optimal monetary and fiscal targets in stabilizing consumption and minimizing employment fluctuations. The maximum sustainable output levels correspond to the minimum equilibrium of unemployment rates. This solution of the stochastic differential equations is the maximum likelihood estimate, and satisfies the coexistence equilibrium under reverse causation. The regression works like simulations. The prediction is the quasi-stationary equilibrium and tends to equal the expected value and can be steered by the optimal sustainable policy.

An open problem of the Nash political noncooperative equilibrium is:

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How can the social welfare be maximized by \( n > 1 \) governments and agents in the monetary union. How is the optimization problem solved at all horizons as \( n \to \infty \)? Governments facing a cyclical recession would prefer a monetary expansion while governments in countries without recession would prefer the zero inflation and oppose the positive inflation\[8\]. The Nash zero equilibrium is solved by weights of votes under politicization. The first-order derivative denotes the weight coefficient. The equilibrium inflation is biased downward.

In our quite startling proof, however, the maximum equilibrium estimate denotes the first-order and the second-order derivative coefficients of optimization and yields the unique solution. Our non-zero equilibrium is unbiased, consistent and robust to the weights and response coefficients as well as shocks. The equilibrium denotes the maximum expected welfare at an infinite horizon. In equilibrium, governments are independent and non-cooperative. In disequilibrium, governments act either positively and cooperatively or act negatively and non-cooperatively. The equilibrium is supported by the optimal policy at no trade-off cost between the policy volatility and stable consumption and employment. It is close to the first best solution for the optimal policy, which alters the initial endowment and exogenous variables, as well as reallocation of taxes, money supply and government spending. We can provide a statistical test to detect the unique institution of optimum equilibrium at the turning point. The convergence probability is the performance criterion for robustness of the optimal policy of taxes and subsidies. The existence of optimal policies is effective and consistent with but is not discovered in team experiments, punishments and incentives as well as in international specialization, where policy can alter the terms of trade [11, 25]. Support for aggregate spending is a V-shaped function of income inequality [24]. The desire of politicians to ‘community-build’ matters more while altruistic or noncooperative motivations matter less in resettled communities [2].

The econometric issue raised is that the volatility of policy can have negative and positive impacts upon output. Contraction of monetary growth has negative impacts upon the unemployment rates in Taiwan and South Korea, but [18] does not detect the optimal monetary growth. International capital inflow tends to reduce the unemployment rate; but [17] provides neither empirical test nor explains why within one year, contractionary monetary policy led to high unemployment rates and low output growth in 2001-2002 in Taiwan and in 1929-33 in the United States. High unemployment rates are passively accepted by workers [20]. Chen [7] simulates the Taiwan
model, uses the calibrated parameters of the U.S. economy; he fails to test how the spatial and intertemporal causal differences, such as a high public investment share and a low public consumption shares, can affect the unemployment. In Africa, the share of public investment is as high as five percent; the share of private investment is as low as six percent. The private investment is crowded out and has no significant positive impacts upon economic growth. In methodology, however, the vector-autoregression (VAR), non-parametric regression, linear regression and bootstrap methods have weak convergence and tend to yield biased estimators [9]. The calibrated parameters of Monte Carlo simulations yield results which vary with constrained performances of models used.

In the present paper, a new method of convergence probability is indicated by the coefficient of determination $R^2$, and used to test the negative or positive impacts of policy volatility, the optimum equilibrium and the optimal policies over business cycles. We show a new theory of the optimal equilibrium in consumption rate, which can be attained and is actually feasible in the absence of commitments. The optimal monetary policy has a stronger convergence probability than the real interest rate rule, which stabilizes the output and inflation at the optimal equilibrium. This equilibrium target lies at the intersection point of supply and demand curves and is robust to the time-varying response coefficients of policies.

We test the hypothesis of parameter estimator $\theta$ about the existence of the non-zero equilibrium $x^*$:

$$H_0 : \theta(x^*) = 0 \text{ versus } H_1 : \theta(x^*) \neq 0$$

A superior technique for hypotheses testing is dynamic quadratic regression which detects the equilibrium for adjusting the policy mix. The dependent variable is the convergent probability $p$, and shows the speed of convergence of the state variables to the optimum equilibrium, which is steered by the counter cyclical policy. The nonlinear systems are solved as a reduced, closed and feedback form, recursively. The equilibrium is stabilized by a set of optimum policies, such as money growth, the real interest rates, taxation, the share of government spending, and budget deficits. The regression works like simulations. We select and optimize the equilibrium through different policy feedback rules. In a previous version, Hsieh [13] finds that equilibrium output levels and output growth are higher than the average, while the optimal inflation rate is 3%, which is lower than the average rate 5%. When output deviates from the equilibrium, the response coefficients are systematic, positive and negative in sign. By convergence probability, the optimal
money growth is more powerful than the interest rate rule. With time-varying coefficients, we compute the optimal equilibrium and policy variables such as the government spending and money supply. In contrast, Mundell [26] resolves policy conflicts with constant coefficients, adjusts the external imbalance through monetary policy, and the internal imbalance through fiscal policy.

1.2 Review of literature

The independent central bank achieves a lower inflation rate but does not reduce unemployment rates [12]. A solution is to cut the central bank’s operation budget, bankers’ term length, and the introduction of more frequent reporting requirements for the congress. The expansionary government spending has a positive impact upon output [4]. Expansionary monetary and taxes also have negative impacts upon output growth. Thus in simulations, a structural model has instable parameter estimates; inflation indexation can increase, decrease, or stabilize the real wage, depending upon productivity or demand shocks [23].

The existing literature ignores the fixed point of the optimal policy. The conflicting theories and evidence exist about the negative or positive impacts of uncertainty and the optimum in debts and deficits [22]. It is assumed that greater income uncertainty leads to greater precautionary saving. Large deficits and high inflation are attributed to non-decentralization, non-democratic institutions, political instability, unprofitable state-owned firms and bad projects, as well as a large share of government spending in GDP. Cumulative deficit tends to raise the real interest rate, reduce private investment and income, increase consumption, and expand the wedge between productivity and wage costs. Thus, unemployment rates soar. Few works have tested convergence probability to the equilibrium or the optimal consistent policy targets, when the nominal wages are rigidly paid due to the long duration of production processes.

1.3 Assumptions

In this study, first, we assume that in political and econometric processes, the sequence of the data of observations is serially correlated, nonstationary, and nonlinear. Second, it is assumed that in equilibrium, the unobserved random shocks of supply and preferences follow the Brownian motions and are independent of the equilibrium and policy, as observations converge to the equilibrium. The solution of optimization is the equilibrium where the expected errors are zero. At disequilibrium, heteroscedasticity of
errors exists; such errors represent misconceptions and bias. Households maximize the wages and minimize the negative impact of variances of policy upon real wages. We test the hypothesis that the first order coefficient is instable and estimate the optimal money growth exists. The money supply can be an exogenous and endogenous variable, and has mutual causality with output.

1.4 Main results

The steady state solution of the maximum welfare is the equilibrium consumption and optimal monetary and fiscal policy. We combine the supply and demand curves of consumption and output into a new nonlinear co-integration model. Households maximize the utility function. We allow the utility to be nonlinear, non-smooth, or not exactly known. Dynamic quadratic regression estimates the optimal policy with a consistent and shrinking confidence interval. In a concave-down utility function, policy can alter the logarithmic output and output growth. Deviations from the optimum have negative impacts and reduce income and labor productivity. The high volatility of money supply reduces the interest rate, whereas a large government spending and deficit raises the interest rate. In the unemployment functions, we find that when the interest rate, government spending, or taxation rises beyond the maximum equilibrium, and when money supply falls below the optimum, policy volatility has positive impacts upon unemployment rates; high interest rates or low money supply raises the unemployment rate. Output declines. Consumption declines. Too high interest rates and too low consumption spending tend to cause ineffective demand, reduces the nominal gross domestic product (GDP), and discourages investment. Beyond the optimum, fiscal expansion and budget deficits can reduce output; conversely, below the optimum, expansionary policy can raise output. If high unemployment rates are accompanied by trade surplus and a high real interest rate, monetary expansion is required. Beyond the optimum, huge public debts and persistent budget deficits tend to raise the real interest rate, reduce savings in the public sector, and increase the unemployment rate.

The optimal policy targets stabilize the maximum equilibrium of the consumption growth rates and consumption shares. The equilibrium under the balanced budget rule is unbiased and consistent, and is attained with the minimum shrinking variance, leading to a low unemployment rate. Our hypothesis-testing is such that a unique optimum equilibrium of output is steered by policy and has the mutual causality or interaction with the policy. The non-zero optimal solution is estimated by dynamic quadratic regression.
In the following, Section 2 shows a new model of the optimum equilibrium where the consumption growth and government spending are optimized interactively. Cumulative budget deficits, however, could reflect bad projects and institutions. It raises the real interest rate and reduces consumption. Beyond the optimum, large budget deficits and hyperinflation are caused by political instability, thus reducing output levels and output growth. Section 3 illustrates empirical examples. Section 4 concludes with remarks.

2. The Model

Notations: Suppose $U: \mathbb{R}^n \rightarrow \mathbb{R}$ is the social welfare or utility, which is a function of consumption. $t \in \mathbb{R}$ is a discrete time or year. \{y, $Y$, i, $\pi$, u, C, $\dot{C}$, b, B, I, G, M, Saving, r, T, w, $\lambda$, P, p, p**, s, v\} $\in \mathbb{R}^n$. $\mathbb{R}^n$ is an n-dimensional Euclidean real space. y is output; $Y \equiv \text{dlog}y/\text{dt} \approx \Delta y/y$ is the economic growth. i is the nominal interest rate; $\pi$ is the inflation rate. u is the unemployment rate. b is the benefits of unemployment, health insurance, or social welfare. B public debts. C consumption; $\dot{C} \equiv \text{dlog}C/\text{dt}$ is consumption growth. I is investment; G is government spending; $E_t$ is the expectation operator based on information up to time t. M is money supply. Saving is private saving; r(t) is the real interest rate on public debts at time t. T is tax revenues. w the aggregate wages. $\lambda$ is the Lagrangian multiplier. Suppose $p = \log P$ is the logarithm of the domestic price level; $p**$ is the logarithm of the foreign price level. s is the logarithm of the nominal exchange rate, i.e., the domestic currency price per unit of foreign currency. Suppose one asterisk denotes the optimum equilibrium. Two asterisks denote the foreign variable or the optimal control policy. v is a vector of perturbations; in disequilibrium, its sample mean of errors may not be zero; and its variance is time-varying. The $\theta$, a and b’s are parameters. F: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a production function. H: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is the Hamiltonian function.

Assumption 1 on nonstationarity: The sequence of random observations are nonstationary, nonlinear, and serially correlated. The fixed point is the general equilibrium in a continuous sense. The nominal wage is rigid.

Assumption 2 on the existence of the unique equilibrium: The serial correlation implies that the optimal policy stabilizes the output growth as well as the output level, because in the nonexpansive, convex, and continuous domain $\Omega$, the fixed-point exists. It is the quasi-stationary output growth and the equilibrium, that is supported by the optimal policy and meets the necessary and sufficient conditions for optimization.
The balanced budget rule implies that policy oscillates and that tax revenues equal the government spending periodically over business cycles:

\[ T = G + v = \frac{u}{1-u} b + v \quad \text{for } u = u^* \text{ and } r = r^* \]  

(2.1)

where government spending consists of unemployment benefit, b, and other payments and stochastic spending, v. The unemployment benefit is the benefit b multiplied by the probability of unemployment, u/(1-u). w is workers’ wages which increase with the after-tax income and unemployment benefits. Wages consist of labor efforts and human capital. Decreases in money supply tend to reduce inflation and increase the real wage and unemployment rates. Workers have to pay taxes for unemployment benefits. Suppose the aggregate wages are close to the after-tax disposable income:

\[ w = (y-(\eta G/y T) + \frac{u}{1-u} b + v) \]  

(2.2)

where w is wages. Taxes are imposed only on the employed workers. \( \eta_{G/y} \) is the elasticity of government spending (for unemployment benefits) with respect to the aggregate income.

To search for the optimal policies, the government maximizes the social welfare, U:

\[ \max_{C \in \mathbb{R}} U(C) \]  

(2.3)

subject to the dynamic state equation

\[ C(t) = F(C(t-1), y(t)) \]  

(2.4)

where consumption is determined by income or wages, and habit persistence of consumption. Government spending on unemployment insurance benefits tends to stabilize consumption and effective demand.

Identities of income and expenditures always hold:

\[ C(t) = y(t) - I(t) - G(t) \]  

(2.5)

\[ y(t) = C(t) + \text{Saving}(t) + T(t) \]  

(2.6)

The budget constraint is

\[ B(t+1) = (1+r(t))B(t) + P(t)(G(t) - T(t)) + \Delta M(t) \]
\[
\begin{align*}
\text{or } & \quad \frac{B(t+1)}{y(t+1)} - \frac{B(t)}{y(t)} = (r - \dot{Y}) \frac{B(t)}{y(t)} + p(t) \frac{G(t+1) - T(t+1)}{y(t+1)} \\
& \quad + \frac{(M(t+1) - M(t))}{y(t+1)} \\
\end{align*}
\]

(2.7)

where Equation (2.5) is an identity of income and expenditure. Equation (2.6) denotes that income equals consumption plus the private saving, and public saving or tax revenues. Equation (2.7) is the budget constraint. As far as the interest rate does not exceed the output growth, \(r = \dot{Y}\), the ratio of debts to income will not increase, because public debts can promote capital formation and increase output and employment rates. If the ratio of debts to GDP increases and cumulates beyond the optimum, however, the government need cut the deficit and its spending to match with tax revenues.

**Definition on convergence probability towards the equilibrium:**
The time invariant solution in probability is the optimum equilibrium, such as \(C(t) = C(t-1) = C^*\) with the convergence probability \(p = \exp(dC/dt) = f(C)\).

\(f: \mathbb{R}^+ \rightarrow \mathbb{R}\). The thrice continuous differentiable function \(f(C)\) indicates the probability distribution of random observations \(C(t)\).

**Proposition 1 on countercyclical policy:** Under a balanced-budget rule, the unemployment rate and changes in tax revenues switch from a negative to a positive relation over business cycles. When \(u(t) < u^*\), then the elasticity of government spending with respect to income is negative, \(\theta \equiv \eta_{GY} < 0\); and vice versa. Thus, the optimum equilibrium of unemployment rate, \(u^*\), is unique; and budgets are balanced by countercyclical policies over a cyclical period.

**Proof:** Suppose that the budget is balanced over business cycles, when output growth rates are equal to its expected average growth. If there exists the unique optimum equilibrium of unemployment rates, the necessary and sufficient conditions are the switching elasticity, \(\theta \equiv \eta_{GY}\), of government spending with respect to income over business cycles:

\[
\begin{align*}
\dot{u}(t) &= u^* + \eta_{GY} (T(t)-G(t)) \quad \Rightarrow \quad \dot{Y}(t) = \dot{Y}^*(t) + \theta(t) (r(t)-r^*) \\
\end{align*}
\]

(2.8)

where \(\frac{\partial T}{\partial u} = -\frac{\partial G}{\partial u} = \frac{(u(t) - u^*)}{1-(u(t) - u^*)} = \eta_{GY} < 0\).
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\begin{align*}
\frac{\partial T}{\partial u} &= - \frac{\partial G}{\partial u} = \eta_{Gt} = 0 \quad \text{for } u(t) = u^* \\
\text{for } \theta < 0 \text{ and } u < u^* \quad (2.9a) \\
\frac{\partial T}{\partial u} &= \frac{\partial G}{\partial u} = \eta_{Gt} > 0 \quad \text{for } \theta > 0 \text{ and } u > u^* \quad (2.10)
\end{align*}

where \( u^* \) is the minimum or lower-bound unemployment rate, when the optimum unemployment is a discipline for workers. \((2.8) \text{ and } (2.9a) \) are the aggregate supply curve for employment if \( \theta(t) < 0 \text{ and } \eta_{Gt}(t) < 0; \) conversely, \((2.10) \) is the aggregate demand curve. \((2.9a) \) implies that budget surplus exists during booms, when \( u < u^*, \eta_{Gt} \geq 0 \text{ and } b > 0. \) The benefit \( b \) is paid for the unemployed. When tax revenues increase, government spending on unemployment benefits decreases. Unemployment rates are low, and inflation rates are high. The optimum unemployment denotes a discipline and can improve the output (and income). In Equation \((2.9b) \), the budget is balanced, when the annual output growth is close to its average growth rate, and the unemployment rate is the equilibrium rate.

In Equation \((2.10) \), during depression when the unemployment rate is high, \( u > u^* \), the unemployment benefits, \( 0 < b < \infty \), are positive and finite for a short period. Government spending provides subsidies for the creation of job vacancies, public investment, and tax credits for private investment. A further increase in unemployment rates tends to reduce tax revenues, \( \eta_{Gt} < 0. \) Budget deficits expand. Q.E.D.

**Proposition 2 on the invariant equilibrium and time-varying coefficients:** When unemployment rates are low, there are budget surplus, or the deficit decreases; and vice versa. The convergence to the unique optimum equilibrium is time-consistent and unbiased. \( \lim_{t \to \infty} p(u(t)) = \exp(du/dt) = \exp \{ \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} [u(t) - u^*] \} = 1. \) 0 < \( t \leq n. \) n is the sample size. If \( u > u^* \) and \( \dot{Y} \)

\( \dot{Y} = E \dot{Y}(t), \) then \( T < T^* \) and \( \Delta G(t) = G(t) - G(t-1) > 0. \) The optimal real interest rate exists. The interest rate rule switches from the supply to the demand side, i.e., from \( \theta_3 \equiv c(t) < 1 \) to \( \theta_4 \equiv c(t) > 1 \) or from \( r < r^* \) to \( r > r^* \) over time; and vice versa.
Proof: To attain the minimum unemployment rate, central banks raise the nominal interest rate in proportion with the inflation rate so as to sustain the optimal real interest rate. At equilibrium, the response coefficient to inflation cost is $\theta_3 \equiv c(t) = 1$:

$$i(t) = r^* + \theta_3(t) \pi(t) + v(t)$$  \hspace{1cm} (2.11)

where below the optimum $r < r^*$ and $\pi > \pi^{**}$, $\theta_3(t) > 1$; and beyond the optimum, $r > r^*$, $\pi < \pi^{**}$, $\theta_3(t) < 1$.

$i$ is the nominal federal fund rate. $\pi$ is the inflation rate. $r = i + \pi$. $r^*$ is the optimal equilibrium in real interest rate. When income and employment rates decrease, the income tax revenues also fall, while government spending increases. It increases budget deficits and effective demand. However, when budget deficits lead to cumulative public debts and increase the real interest rate, large deficits discourage private investment and raises the unemployment rates. The government must cut its spending to match with tax revenues.

The optimal fiscal policy is countercyclical around the maximum equilibrium growth $Y^*$. The first and the second-order derivatives of government spending $G(Y)$ are:

$$\frac{d\log G}{dt} = \theta_0 + \theta_1 \dot{Y} + \theta_2 \dot{Y}^2$$

$\theta_1 > 0$ and $\theta_2 < 0$

$$= \theta_2 (\dot{Y}(t) - \dot{Y}^*)^2 \quad \text{as} \ G \rightarrow G^* \ y \rightarrow y^*,$$  \hspace{1cm} (2.12)

where $\dot{Y}^* \approx (\Delta y^*/y^*) = -\theta_1 / 2 \theta_2$ is the quasi-stationary maximum output growth. During booms, when the output growth equals or exceeds the maximum equilibrium $Y(t) = Y(t-1) = Y^*$, the growth rate of government spending declines to zero. During the booms when tax revenues exceed the government spending, budget surplus is used for repaying for cumulated public debts and stabilizing the real interest rate. Budgets are balanced over cycles, say, every six years. The attracting target is a set of the optimum equilibrium $\{u^*, C^*, Y^*\} \in L_2$. $L_2$ is the Hilbert space of the least squares vector-integrated functions. Tax-cutting and expansionary fiscal policy are recommended during depression rather than booms. Q.E.D.

From (2.2) and (2.3), in equilibrium, the Hamiltonian function is rewritten as
H(C, y, \lambda) = U(C) + \lambda (C - F(C, y)) \quad (2.13)

**Remark 1 on the fixed point:** We will solve the fixed point for Equations (2.3), (2.4) and (2.13). We solve Equation (2.13) which is Lucas’[11] unsolved model \( H = U(C) - C \). One additional unit of output corresponds to one additional unit of consumption, labor effort, and capital formation. Households search for the optimal labor effort. For example, our optimal policy of low positive inflation tends to not only reduce the government’s cost of tax collections, stimulate private investment of the firms, but also reduce labor income taxes, and stimulate labor effort.

**Proposition 3 on minimum variance:** The optimal consistent policy \( r^{*} = r^{*} \) increases the convergence probability, \( p \), towards the equilibrium,

\[
\lim_{t \to \infty} p = \lim_{t \to \infty} \exp (dC/dt) = \exp \left( f(C, \gamma) = 1 \right),
\]

where \( dC/dt \approx f((C - C^*)/\sigma_c)^2 \)

\[
\approx \theta_2 (C - C^*) dt + \theta_4 \sigma_c dv \quad (2.14b)
\]

The equilibrium \( C^* \) has the minimum and shrinking variance:

\[
\frac{1}{n} \sum_{e=1}^{n} dC/dt = \frac{1}{n} \sum_{e=1}^{n} \theta_2 (C - C^*)^2 < \frac{1}{n} \sum_{e=1}^{n} \theta_2 (\tilde{C} - C)^2
\]

if \( C^* \neq \tilde{C} \) \quad (2.14c)

Unlike the noncooperative Nash zero equilibrium, (2.14c) implies that the non-zero equilibrium \( C^* \) tends to be acceptable for all participants in a game problem. \( \sigma^2_c = \lim_{t \to \infty} \frac{1}{n} \sum_{t=1}^{n} (C(t) - C^*)^2 \) is the variance of consumption. (2.14) is a stochastic differential equation. The variance \( \sigma^2_c \) of consumption \( C \) will be controlled by the variance of interest rates \( r \). As \( r \to r^{**} \), \( C(t) \to C^* \). The equilibrium \( C^* \) has the minimum variance.

**Proof:** The policy variable, such as the real interest rate, is an implicit cost of resources, and serves as a proxy for the Lagrangian multiplier \( \lambda \). \( v \) is stochastic errors. To stabilize consumption, the fiscal and monetary policies steer the consumption to the equilibrium.

Equation (2.13) is solved as the canonical Hamiltonian equations:

\[
\partial H(C, y, \lambda) / \partial C(t) = \partial U(C(t)) / \partial C(t) - \lambda(t) = 0
\]
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\[ (\lambda \cdot t) = 0 \quad \text{if } U(C(t)) = \exp(-\lambda t) \log C \]

To find the time-consistency policy, we solve the second-order derivative:

\[
\frac{dC}{dt} = \frac{\partial}{\partial \lambda} H(C, y, \lambda) \tag{2.15a}
\]

\[
\frac{dy}{dt} = \frac{\partial}{\partial \lambda} H(C, y, \lambda) = f(Y, C, \lambda) \tag{2.15b}
\]

\[
\frac{d\lambda}{dt} = -\frac{\partial}{\partial y} H(C, y, \lambda) \tag{2.15c}
\]

where \( r \equiv \lambda \) serves as a proxy of the opportunity cost of resources, \( \lambda \).

From equations (2.15a) through (2.15c), we take partial derivatives and yield the equilibrium:

\[
\frac{d}{dt} H(t, C, y, \lambda) = \frac{dC}{dt} + \frac{\partial H}{\partial C} \frac{\partial C}{\partial r} \frac{\partial r}{\partial t} = 0 \quad (2.15d)
\]

\[
\frac{d}{dr} H(t, C, y, \lambda) = \frac{dr}{dt} + \frac{\partial H}{\partial C} \frac{\partial C}{\partial (G-T)} \frac{\partial (G-T)}{\partial r} \frac{\partial r}{\partial t} = 0 \quad \text{for } r \equiv \lambda \tag{2.15e}
\]

where in (2.14a), the convergence probability is the p value, and reflects the possible deviation of the state variable, C, from the optimum equilibrium, \( C(t) = C(t-1) = C^* \); and the variance tends to be zero. The time-dependent Hamilton-Jacobi equation is

\[
\frac{dC}{dt} + H(r, t, C, dC/dr) = 0 \tag{2.15f}
\]

which is refined as a generalized solution

\[
\frac{dC}{dt} + \Theta_1 \frac{\partial H}{\partial C} + \Theta_2 \frac{\partial^2 H}{\partial C^2} + \Theta_3 \frac{\partial H}{\partial (G-T)} + \Theta_4 \frac{\partial^2 H}{\partial r^2} + \Theta_5 \frac{\partial H}{\partial r} \frac{\partial r}{\partial t} = 0 \tag{2.16}
\]

where at equilibrium, \( \frac{dC}{dt} = f(C^*, r^*) = 0 \). The first and the second-order derivatives are approximated by a strictly concave function, \( f(.) \), or the quadratic regression equations. The econometric process is as follows:

**Step 1**: Estimate the dynamic quadratic regression
\[ C = \theta_0 + \theta_1 C(t-1) + \theta_2 \Delta C(t-1)^2 + \theta_3 r(t-1) + \theta_4 r^2(t-1) + v(t) \]  

(2.17)

or

\[ - \theta_1 C(t-1) = \theta_0 + \theta_2 C(t-1)^2 + \theta_3 r(t-1) + \theta_4 r^2(t-1) + v(t) \]

if \( \theta_1 > 0 \) and \( \theta_3 < 0 \)

where as in Figure 1, the vertical axis denotes the state variable; the horizontal axis denotes the control variables. At equilibrium, \( C = C(t) - C(t-1) = 0 \).

\[ \frac{\partial C}{\partial r} = - \frac{\theta_1}{\theta_3} > 0 \] if \( r < r^{**} \) and \( C < C^* \); and

\[ \frac{\partial C}{\partial r} < 0 \] if \( r > r^{**} \) and \( C > C^* \).

Similarly,

\[ \Delta r(t) = b_2 ((r(t) - r^*)^2 + b_4 (G/y - (G/y)^*)^2 + b_6 \{ (T/y - (T/y)^*)^2 + v^2(t) \} \]  

(2.18)

d\log C^*/dt and C*/y

The share of Consumption to GDP

The Minmax Solution

(Concavity down)

\( \theta_1 > 0 \)

\( \theta_4 > 0 \)

\( \theta_3 < 0 \)

0% (T/y)** Minimum Income Tax Rate

(G/y)** Maximum Share of Government Spending

r** Maximum Interest rates( \( \leq \) output growth)

short-run \( t < t^* \)

long run \( t > t^* \)

t^* time

Figure 1 Maximum Consumption Rate, C/y and d\log C^*/dt

where, using quarterly data, we add the delay impact through the autoregressive terms, such as \( \Delta r(t-1) \) and \( \Delta r(t-2) \). \( r \) denotes the real interest rate. \( v \) is the residuals with a zero mean and a constant variance. In Equation (2.17), the consumption (or income) is stabilized through the interest rate, and as in Equation (2.18), indirectly through the balanced budget.

**Step 2:** Estimate the equilibrium and the optimal policy and reduce heteroscedasticity.
\[ C(t) = \theta_2 (C(t-1)-C^*)^2 + \theta_4 (r(t-1)-r^{**})^2 + \theta_5 (r(t-1)-r^{**})^2 (C(t-1)-C^*) \]
\[ + \theta_6 C(t-1) + \theta_7 C(t-2) + \nu_1(t) \]  
(2.19)

where the last interactive terms in Equation (2.19) is used to minimize the heteroskedasticity of errors. \( C = \Delta \log C = \log C(t) - \log C(t-1) = \Delta (t/C(t)). \)

\[ C(t) = \theta_5 (r-r^{**})^2 (C(t-1)-C^*) \] for \( \theta_5 <0 \) if \( C(t-1) > C^* \) and \( \theta_5 >0 \) if \( C(t-1) < C^* \)

where the stability of the equilibrium \( C^* \) requires changes in policy responses \( \theta_5 \).

\[ \theta_2 C(t-1) + \theta_7 C(t-2) \approx C(t-t*) \] for \( t* = \theta_6/2(\theta_7) \)

where if \( \theta_6 (C(t)-C(t*)) = \theta_7 C(t-t*) \), then \( \left( C(t-t*)^2(t-t*)^2 \right) = \left( C(t-t*)^2(t-t*)^2 \right) \).

Time \( t* \) is periodical of the period of policy delay impacts. Q.E.D.

Equation (2.19) yields the probability \( p(x) \) against deviations of any of \( n \) observations from the stable equilibrium:

**Corollary 1 on optimum equilibrium and policy:** If \( \theta_1 >0 \) and \( \theta_2 <0 \), \( C^* = \theta_1/2 \theta_2 \) is the maximum equilibrium at the stopping time \( t* \).

\( \frac{dC}{dt} = \theta_2 (C(t)-C^*)^2 <0 \) if \( C(t) > C^* \), implying that the volatility cost of deviations tends to reduce consumption. If \( \theta_1 <0 \) and \( \theta_2 >0 \), \( C^* \) is the minimum at the ramification point \( t* \). If \( \theta_1 >0 \) and \( \theta_2 >0 \), \( C^* \) is the unique equilibrium and the inflection point. \( \theta_3 >0 \) and \( \theta_4 <0 \) denotes that \( r^{**} = -\theta_3/2 \theta_4 \) is the maximum policy. As \( \lim_{t \to \infty} \Delta C = \lim_{t \to \infty} \frac{dC}{dt} = 0 \), the feedback rule, \( r=r^{**} \), leads to the maximum equilibrium, \( C(t)=C^* \). According to the law of large numbers, the process \{C(t)\} has a spherical convergence towards a fixed point \( C^* \) in a finite time \( 0 < t \leq n < \infty \).

\( \sigma_r^2 = \lim_{t \to \infty} \frac{1}{n} \sum_{n=1}^{\infty} (r(t)-r^{**})^2 \) is the time-consistent estimate of the variances of interest rate policy which stabilizes output. \( \lim_{t \to \infty} \exp(\Delta C) = 1 \)
denotes the convergence probability close to one, as \( \{C, r\} \rightarrow (C^*, r^*) \).

**Corollary 2: The convergence probability to the maximum equilibrium:**

Suppose the welfare function is \( F(t)=u(t)x(t) \). The first derivative of \( F \) is \( \frac{dF}{dt} = \left( \frac{du}{dt} \right) x(t) + u(t) \left( \frac{dx}{dt} \right) \geq 0 \). To test the existence and the uniqueness of the stable equilibrium at all horizons, the convergence probability is \( \lim_{t \rightarrow \infty} p(x) = \exp(\frac{dx}{dt}) = f(x) = 1 \) as follows:

\[
\lim_{t \rightarrow \infty} \frac{dF}{dt} = \lim_{t \rightarrow \infty} \frac{dx}{dt} = \frac{dx}{dt} = 0
\]

where the solution converges \( x(t) = x(t-1) = \ldots = x^* \) before the variances of policy go to infinity, \( \sigma_u^2 = \sigma_v^2 = \frac{1}{n} \sum_{i=1}^{n} (u(t)-u^*)^2 \) as \( n \rightarrow \infty \).

**Remark 2:** Antipin’s et al. [1] solution is not computable without iterative errors:

\[
\lim_{t \rightarrow \infty} \frac{dF}{dt} = \lim_{t \rightarrow \infty} \frac{dx}{dt} = \frac{dx}{dt} = 0
\]

With no iteration, our maximum equilibrium always exists and is non-empty; it is acceptable to \( n \) governments and all participants, as \( n \rightarrow \infty \), and robust to the cooperative and non-cooperative games and the problems unrelated to games, as well as the problem of acceptance or rejection of proposals.

We use Equation (2.18) to reduce the sensitivity of the Student \( t \), \( \chi^2 \) and \( F \) tests, which otherwise are not robust to heteroskedasticity and abnormality of residuals. \( t^2 = \chi^2 (1) = \frac{1}{n} \sum_{i=1}^{n} (C(t)-C^*)^2 \) follows the Student \( t \) and the chi-squares statistic with one degree of freedom as the observations \( C(t) \) converge to the equilibrium with \( \lim_{t \rightarrow \infty} \sigma_c^2 = 1 \). Q.E.D.

**Remark 3 on the impact, negative \( \theta_4 < 0 \), or positive \( \theta_4 > 0 \), of volatility of policy:** In Equations (2.17) and (2.18), in the concave-down utility function \( f \), the volatility has a negative impact \( \theta_4 \) upon consumption, \( C(t)-C(t-1) \leq 0 \) if \( C(t-1) > C^* \). In the convex-up loss function \( f \), the volatility has a positive impact upon \( C(t) \). Under the balanced budget rule, the optimal
policy is countercyclical. It stabilizes the optimum equilibrium of output and unemployment. Budget deficits and cumulative debts during depression are repaid through budget surplus during the booms. Such a solution sustains the optimal equilibrium \((C^*, r^{**})\) of consumption and the real interest rate.

**Proposition 4 on policy bias**: The optimum equilibrium about the state variable is a Nash-Stackelberg equilibrium as if the government is a Stackelberg leader. In the non-expansive compact convex domain, the equilibrium of unemployment rate \(u^*\) always exists and is time-consistent and unbiased. It is actually feasible in the absence of commitments.

\[
\lim_{t \to \infty} \frac{1}{n} \sum_{t=1}^{n} \frac{du}{dt} = \lim_{t \to \infty} \frac{1}{n} \sum_{t=1}^{n} \theta (u(t) - u^*)^2 = 0.
\]

The deviation from \(u^* \neq u^{**}\) denotes inflation bias, fiscal policy bias and existence of limit cycles.

**Proof**: The equilibrium exists regardless of whether the central banks are independent or committed, or whether or not the policies are coordinated. Its existence is verified by its turning point and its second-order condition. For example, \(\partial f(Y, r)/\partial r < 0\) for \(r < r^*\); and \(\partial^2 f(Y, r)/\partial r^2 \geq 0\) for \(r \geq r^*\), where at least one inequality exists. The necessary and sufficient conditions of optimization indicate that the extreme policy without constraints is not optimal. The welfare \(H(.)\) is Pareto-improving and maximized through a sustainable balanced budget and interest rates as follows:

\[
H(C(0), Y(0), r(0)) \leq ... \leq H(C(t-1), Y(t-1), r(t-1)) \leq H(C(t), Y(t), r(t)) \leq H(C(t+1), Y(t+1), r(t+1)) \geq ... \forall t \in \mathbb{R}
\]

subject to the balanced-budget constraints:

\[
\begin{align*}
\text{r(t-2)} & \leq \text{r(t-1)} \leq \text{r(t)} \leq \text{r}^* \leq \text{r(t+1)} \leq ... \\
\text{(G(t)-T(t))/y(t)} & \leq \text{(G(t)-T(t))/y(t)} \leq \text{(G^*-T^*)/y^*} \leq \text{(G(t+1)-T(t+1))/y(t+1)} \leq ...
\end{align*}
\]

where at least one strict inequality holds. In contrast, Glazer et al.[10] and Roemer [27] suggested that extreme policy increases the gains from bargaining. They neglected the limitation, the stability, and the attainability of the global equilibrium.

Thus, the simultaneous model is closed through an augmented Okun law:
\[ \Delta \dot{Y}(t) = f(\dot{Y}(t), u) + \nu_6(t) \]

or

\[ \Delta \dot{Y}(t) = -\theta_2 (Y(t) - Y^*)^2 - \theta_4 (u(t) - u^*)^2 + \nu_6(t) \]

where in Equation (2.20), output growth is the state variable, which is stabilized by the unemployment rate as a control variable.

The augmented Phillips curve:

\[ \Delta u(t) = a_2 (u(t) - u^*)^2 + a_4 (r(t) - r^*)^2 + a_6 (w(t) - w^*)^2 + \nu_5(t) \]

(2.21)

where \( u \) is the unemployment rates. \( w \) is the real wage growth. In Equation (2.21), as the unemployment rate rises, output growth falls. In Equation (2.21), if \( a_2 > 0 \), \( u^* \) is the minimum unemployment rate, which corresponds to the maximum output growth \( Y^* \). The unemployment rate is a state variable. If \( a_4 > 0 \), a further rise in high interest rates tend to increase the unemployment rate \( u^* \) which is the minimum; \( s^{**} \) is the minimum interest rate. As \( a_6 > 0 \), \( w^* \) is the minimum wage growth. A rise in wages tends to increase the unemployment rate.

**Remark 4 on the unbiasedness under the floating exchange rate:**

Using Equations (2.17), (2.18) and (2.19), we test for fiscal bias and inflation bias, while reducing the heteroskedasticity, bias, and inconsistency of estimates of coefficients and residuals. For a unit-root specification test of random walks, (2.22) and (2.23), we add a convergence probability (2.24).

The real exchange rate \( q \) is equal to the purchasing power parity, \( q = S(i)P^*(i)/P(i) = 1 \) and is consistent with the nominal exchange rate, \( \log S = s = p - p^{**} \) or \( s = p \) if \( \log P^{**} = p^{**} = 0 \):

\[ \Delta s(i) = a_0 + a_1 \Delta p(i) - a_2 \Delta p^{**}(i) + \nu_1(i) \]

(2.22)

\[ s(i) = b_0 + b_1 p(i) - b_2 p^{**}(i) + \nu_2(i) \]

(2.23)

\[ \Delta S(i) = \theta_0 S(i) + \theta_2 S^2(i) + \nu_3(i) \]

\[ = \theta_2 (S(i) - S^*(i))^2 + \theta_4 (SP^{**}(i)/P(i) - 1)^2 + \nu_3(i) \]

(2.24)

for \( E(\nu_3(i)) = 0 \)

where the null hypothesis is \( H_0 : a_1 = a_2 = b_1 = b_2 = 1 \) and \( H_0 : a_0 = b_0 = 0 \).

0 < i < n, n is the sample size. \( p \) is the domestic price; \( p^{**} \) is the foreign price.

At disequilibrium, Equations (2.22) and (2.23), however, do not yield the i.i.d. residuals with homoskedasticity, normality, and unbiasedness. From (2.24) and (2.19), the prediction tends to be the equilibrium exchange rate.
Corollary 3 on no intervention and no bias: The central banks control the optimal money growth and the real interest rate, and let the foreign exchange rate be floating, and the trade be balanced in lag adjustment. The real interest rate parity provides a real anchor:

\[ i(t) = r(t) + \pi(t) \]

where the nominal interest rate \( i(t) \) equals the real interest rate \( r(t) \) plus the expected inflation \( \pi(t) \). The domestic interest rate equals the foreign nominal interest rate, \( i^*(t) \), plus the expected rate of changes in foreign exchange rate, \( dS/dt \).

**Proof:** With the floating exchange rate, as in [14, 15, 16], the optimal real interest rate is consistent with the purchasing power parity and the maximum output growth. Suppose the foreign exchange rate is fixed. A relative high foreign interest rate tends to induce capital outflow. It causes a domestic credit crunch and bankruptcies. Monetary expansion tends to reduce the domestic interest rate and leads to devaluation of domestic currency. As in [19], banking and currency crises can be a by-product of government budget deficits. Budget deficits can cause a domestic high rate of interest. It reduces effective demand and employment rates. In contrast, [6] suggested to intervene within the target zone of foreign exchange rates. Q.E.D.

Proposition 5 on superconvergence probability \( p=\exp(\Delta u) \) within a finite time \( n<\infty \): The optimal deficit policy \( ((G-T)/y)^{**} \) is used to minimize the unemployment rate \( u^* \), minimize and insulate noises \( v \). As \( (G-T)/y \rightarrow ((G-T)/y)^{**} \), \( u \rightarrow u^* \) within a finite time \( t \leq t^* < n < \infty \). The expected errors are zero, \( E(v)=0 \). Thus one equation is solved for one unknown \( u^* \).

**Proof:** The optimal balanced budget rule leads to no inflation bias and fiscal bias, and results in the unique optimum equilibrium \( u^* \) of unemployment rates over business cycles. The optimal share of deficit in GDP steers the unemployment rate towards the equilibrium \( u^* \). The reduced form of the model (2.1) through (2.24) is

\[ \Delta u = b_2 ((u(t)-u^*))^2 + b_4 ((G-T)/y-(G-T)/y)^{**2} \]

where \( \Delta u = u(t)-u(t-1) \). The unemployment rate is stabilized by the balanced budget over business cycles. As in Equations (2.17) and (2.24), \( b_4 > 0 \).
implies that a high budget deficit (G-T) tends to raise the real interest rate and increase the unemployment rate. If \( b_2 > 0 \), \( u^* \) is the minimum equilibrium of unemployment rates. Q.E.D.

**Remark 5 on time-varying policy and the stable equilibrium**: The optimum equilibrium of the state variables is a set \((\hat{C}^*, y^*, \hat{Y}^*, u^*) \in L_2\) in (2.16)-(2.24). \( L_2 \) is a least square space and a subset of the Hilbert space. In the absence of commitments, the solution is unique. The first best solution, \((\hat{C}^*, y^*, \hat{Y}^*, u^*)\), is stabilized by the countercyclical policy \((r^{**}, (G-T)/y)^{**}; (G/y)^{**}; (T/y)^{**}\). With convergence probability one, \(\lim_{t\to\infty} \exp(du/dt)=f(x^*)=1\), this optimum solution is time-consistent and unbiased. It detects the direction of fiscal bias and inflation bias and other distortions, when the central banks’ intervention in the exchange market is inconsistent with the arbitrage and involves insufficiently contractionary or expansionary policies.

3. Preliminary Examples

In the following, the econometric process is based on the Taiwan quarterly data for 72 periods from 1983.1 through 2001.1, and 144 periods from 1966.1 to 2001.4, as published by the auditing and statistic bureau, Taiwan Government. For illustration purposes, some Student t statistics are omitted.

**Problem formulation on trade-off between optimal inflation rates and unemployment**: We maximize the Hamiltonian function \( H \):

\[
\max_{x \in \Omega} H(x, u, \theta(t))
\]

where \(\{x(t), u(t)\}\) is a sequence of observations. \( \theta \) denotes parameters and multipliers. By the normal reduced form, we reduce the problem in question to a reduced form, which is an equation with fewer unknowns. The sufficient optimality condition is at least the weak convergence, as \( t \to \infty \), \( u \to u^{**}, x \to x^*, \) and \( H(x(t), u(t)) \to H(x^*, u^{**}) \). Symbol \( \to \) denotes convergence. The welfare is Pareto-improving:

\[
H(x^*, t^*)-H(x, t) \geq 0 \quad \text{if} \quad u(t)=u^{**} \geq \theta_2 (x(t)-x^*)^2
\]

Like a heat equation, the law of motion with the second-order suff-
icient condition is
\[ \frac{dH(x)}{dx} = \frac{d^2H(x)}{dx^2} + \nu(t) \]
or
\[ \frac{dH(x)}{dx} = H(x^*) - H(x) \geq 2 \theta (x-x^*)^2 \]
The Lipschitzian continuity of control implies that the variance of the equilibrium is bounded by policy:
\[ \left( \frac{\partial H(x,u,t)}{\partial x} - \frac{\partial H(x,u)}{\partial x} \right)^2 = \theta (x(t-1)-x^*)^2 \leq \theta_2 (u(t-1)-u^*)^2 \]
where \( \nu \) is residual errors. \( \theta_2 \) and \( \theta_4 \) are coefficients. Suppose deviations from the equilibrium \( H(x^*(u(t))) = x^* \) denote a welfare loss.

**Corollary 4:** the marginal utility, \( dH/dx > 0 \), is Pareto-improving with the monotonely increasing sequence of convergence, as \( x(t-1) < x(t) < x^* \), \( H(x(t-1)) < H(x(t)) < H(x^*) \):

The marginal utility is the price of policy option or the convergence probability.
\[ p(x) = \exp(dx/dt) = \exp(dH(x)/dx) = f(x,u) = \theta_2 (x-x^*)^2 + \theta_4 (u-u^*)^2 + \nu(t) \]

Without iterations, the solution of the Cauchy sequence problem is backward and forward consistent:
\[ Y(t) = Y(0) \exp(x(t)) \quad \text{for} \quad x \leq x^* \text{ and } t>0 \]

**Example 1:** Optimal monetary policy and inflation in Taiwan

*Step 1:* The Euler equation is estimated as dynamic quadratic regression:
\[ p(x) = \exp(\Delta x(t)) = f(x, \ \Delta P(t)/P(t-1), (\Delta M(t)/M(t-1))) \]
\[ \Delta x(t) = -102.11 + 10.30x(t-1) + 0.62x^2(t-1) + 17.25 (\Delta P(t)/P(t-1)) \]
\[ \quad + 3.24(\Delta P(t)/P(t-1))^2 + 2.00(\Delta M(t)/M(t-1))-0.108(\Delta M(t)/M(t-1))+\nu(t) \]
\[ (\ -1.94\ ) \quad (\ -2.35\ ) \quad (\ -2.45\ ) \quad (\ -2.45\ ) \quad (\ -2.11\ ) \]
\[ (\ 1.74\ ) \quad (\ -1.94\ ) \quad (\ -1.94\ ) \quad (\ -4.23\ ) \]

\( R^2 = 0.71; \ \bar{R} = 0.69; \ n=76; \ D.W.=0.90; \ 1^{st} \text{order autocor- relation } = 0.54 \)
where the values in parentheses are t statistics. \( p(x) \) is the convergence
probability; \( x \) is output growth; \( \Delta P(t)/P \) is the inflation rate; and \( \Delta P(t)/P \) is the growth rate of money supply (M1). The high and stable economic growth is the attractor \( x^* \approx 9\% \approx 10/2(0.6) \), and denotes the unique reflection point. The optimal inflation rate is \( 3\% \approx 17/2(3) \); the optimal money growth (M1B) is \( 9\% \approx 2.0/2(0.108) \).

Step 2: Estimate the equilibrium from dynamic quadratic regression, and by adding the second-order autoregressive terms, this refinement reduces heteroscedasticity:

\[
\Delta x(t) = 19.5 + 0.63 \Delta x(t-1) - 0.413 \Delta x(t-2) - 0.089(\Delta M/M(t)) - 9%)^2 \\
+ 1.07(x(t)-9%)^2+0.013(x(t)-9%)^2(\Delta M/M(t))-9%)^2 \\
(2.98) (5.73) (-4.03) (-3.02) \\
(5.20) (3.10) (3.2)
\]

\( R^2 = 0.71; \bar{R} = 0.69; n = 76; D.W. = 2.20; 1^{\text{st}} \text{order autocorrelation} = -0.10. \)

In Table 1, as far as the inflation rate stays below the optimum inflation 3\%, as the money supply decreases, output declines. Money growth, inflation, and output growth are positively related.

**Table 1 Indicators in Taiwan over the period 1959.01 to 2001.04**

<table>
<thead>
<tr>
<th>The variables</th>
<th>Sample mean</th>
<th>The standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean real output growth</td>
<td>( \bar{x} = 8.26 )</td>
<td>5.89</td>
</tr>
<tr>
<td>The equilibrium growth ( x^* )</td>
<td>( x^* = 9% )</td>
<td>4.3%</td>
</tr>
<tr>
<td>Mean inflation of consumer prices, ( \Delta P/P(t) = 5.28% )</td>
<td>8.5%</td>
<td></td>
</tr>
<tr>
<td>Optimal inflation</td>
<td>( \Delta P/P(t) = 3% )</td>
<td>1.5%</td>
</tr>
<tr>
<td>Mean money growth(M1B)</td>
<td>( \Delta M/M(t) = 13% )</td>
<td>12.8%</td>
</tr>
<tr>
<td>Optimal money growth(M1B)</td>
<td>( (\Delta M/M(t)) = 9% )</td>
<td>4.5%</td>
</tr>
</tbody>
</table>

*Source: Data come from the Taiwan government statistic department. These values are estimated by the author and research assistants. Output is the gross domestic product. The nominal interest rate is the discount interest rate of the central bank. The inflation rate is the consumers’ price inflation.

Note 1: The standard deviation of the optimal policy denotes the standard deviation of the estimates of first-order parameters of linear regressions and corresponds to the t statistic. In 2001-2002 in Taiwan, within less than two years, when the money growth (M1B) declined by 4\%, the inflation rate fell close to zero. The output growth also fell to zero and decreased by about 4%. 

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We use the following examples to illustrate that when the policy is inconsistent, the coefficient of determination $\hat{R}^2$ under regression also becomes insignificant to explain the variations in the state variables. Accordingly, the convergence probability falls.

**Example 2.1** High real interest rates have positive impacts upon the unemployment rates.

Consider the unemployment and real interest rates in Taiwan. Beyond the optimum, a further rise in interest rates tends to raise the unemployment rate. Let $u$ be the unemployment rate. $r$ is the real interest rate. $0 \leq t \leq n$ is time or quarterly. $n$ is the sample size and a positive integer. $\Delta u=u(t)-u(t-1)$. Using the Taiwanese quarterly data over the period 1977.4 through 2001.4, the trade-off between unemployment and inflation is more significant than between unemployment and the real interest rate. In other words, the equilibrium unemployment $u^*=2\%$ is attained through the real interest rate $r^*=2\%$. The equilibrium economic growth is $x=5\%$.

\[
\Delta u = 0.04 + 0.0006(u(t-1)-2\%)^2 + 0.003(r(t-1)-2\%)^2 + 0.006(u(t-1)-2\%)(r(t-1)-2\%)^2
\]

$(-0.97)$ $ (3.19)$ $(1.47)$ $(2.07)$ $(3.3)$

$R^2 = 0.11$ $ \hat{R}^2 = 0.11$; D.W. = 2.2; $n=92$; 1st order autocorrelation $\rho = -0.139$

where the values in the underlying parentheses are Student t statistic. Deviations from the optimal interest rate $r^*=2\%$ tends to raise the unemployment rate.

**Example 2.2:** Beyond the upper limit $r^*=2\%$, high interest rates have negative impacts upon output.

There exist the optimal interest rate and inflation rate under a demand and supply (bifurcation) model of output growth. Let $x$ be the growth rate of output (GDP). $\pi$ is the inflation rate of the GDP price deflator. $r$ is the real interest rate and equals the central bank’s discount rate minus the inflation rate:

\[
\Delta x(t) = -0.004 - 0.86(x(t) - 5\%)^2 - 0.0002(r(t)-2\%)^2
\]

$(-1.73)$ $(2.64)$ $(-1.77)$

$R^2 = 0.19$ $ \hat{R}^2 = 0.05$; D.W. = 1.6; $n=92$; 1st order autocorrelation $\rho = 0.17$
Example 3: On determinants of unemployment rates. We find the coexistence of the positive and the negative impacts of government spending with a turning point.

According to neo-classical theory, the unemployment rate is assumed to increase, as the government spending crowds out private investment and consumption. According to the Keynesian economics, when government spending increases the effective demand, the unemployment rate decreases. In the linear regression of unemployment rates, the coefficient of direct taxes is significantly negative although the theory predicts it to be positive [22]. The decline in the growth in real government expenditure is responsible for an almost 2 percentage point of growth in unemployment. Increases in government spending and taxation are often assumed to reduce the after-tax wage and increase the unemployment rate.

The law of motion of the welfare with respect to the unemployment rate is estimated as

\[ \frac{du}{d(t)} = f(\theta_1 u, \theta_2 u^2, \theta_3 \frac{d}{d} \frac{u}{G/Y}, \theta_4 \frac{d}{d} \frac{u}{(G/Y)^2}, \theta_5 \Delta u(t-1), \theta_6 \Delta u(t-2), v) \]

where \( \theta_1 < 0 \) and \( \theta_2 > 0 \) implies that the equilibrium unemployment rate \( u^* \) is minimal. \( \theta_3 > 0 \) and \( \theta_4 < 0 \) implies that the maximum share of government spending in national income exists. The autoregressive terms, \( \Delta u(t-i) \) for \( i=1,2,... \), denote the wave-like or cyclical behavior. \( v \) is stochastic errors or sources and sink of disturbances.

In Table 2, we find that both positive and negative impacts of government spending exist. For a small open economy, such as that of Taiwan, the maximum share of government spending in national income could increase to 40% (=G/Y). The sample mean of the government spending in this period would be 28%. The larger the share of trade volumes (or exports and imports) in national income, the larger the government spending would tend to be. We accept the coexistence of the monetarists and the Keynesian theory and accept the hypothesis of the existence of a unique general equilibrium with the convergence probability, \( R^2 = 85\% \).
Table 2 Determinants of Unemployment Rates

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>Unemployment $\Delta u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-1.31 (-2.70)</td>
</tr>
<tr>
<td>$(u(t)-1%)^2$</td>
<td>0.05 (5.09)</td>
</tr>
<tr>
<td>$G/y$</td>
<td>6.60 (3.01)</td>
</tr>
<tr>
<td>$(G/y)^2$</td>
<td>-8.32 (-2.81)</td>
</tr>
<tr>
<td>$\Delta u(t-1)$</td>
<td>1.05 (10.11)</td>
</tr>
<tr>
<td>$\Delta u(t-2)$</td>
<td>-0.49 (-5.18)</td>
</tr>
</tbody>
</table>

$R^2 = 0.86; R^2 = 0.85; n=76; \text{Root MSE}=0.21$

Durbin Watson $D = 2.13; 1^{st}$ order autocor $\rho = -0.13$

Note 1: The values in parentheses are t statistics. $u$ is the unemployment rate; $G$ is government spending; $y$ is national income. $t$ is time or quarters. The average unemployment is $u=1\%$ over the period 1977-2001.

Note 2: The maximum share of government spending (and cumulative debt) in national income is $G/Y = 6.60/(2)(-8.32) = 40\%$. The sample mean of the unemployment rate is 1.2\% in this period.

Example 4: Sensitivity tests on the output growth and optimal inflation rate in Taiwan

The optimal inflation is a bifurcation point. Below the optimum, low inflation ($\pi <1\%$) is found to significantly reduce output growth. Suppose $x$ is the growth rate of output (GDP) in Taiwan over 1995.1 to 2000.1. Suppose that when the inflation rate is less than 1\% ($\pi <1\%$), the dummy variable is $a=0$; otherwise, $a=1$.

$$\Delta x = -4.1 - 0.97(1-a)(\pi(t-1)<1.0\%)+0.30a(\pi(t-1)>1.0\%)+3.40x \quad (3.5)$$

$R^2=0.58; R^2=0.49; D.W.=2.35; n=20; 1^{st}$ order autocorrelation $\rho=0.003$.

The optimal inflation rate is neither too high nor too low, $1\% < \pi^*(t) <3\%$. Deviations from the optimum have a negative impact upon economic growth $x(t)$, and raise the unemployment rates $u$. If the central bank will exercise the policy option, the strike price is the optimal nominal interest rate. The nominal loan interest rates in Taiwan are relatively rigid. The optimal inflation rate is around $d\log P/dt=3\%$. The central bank should switch the optimal discount nominal interest rate around $i=r^*+d\log P/dt$, or
i=2\%+3\%=5\%. Here r** is the optimal real interest rate.

Remark 6 on superconvergence time \( p(x) = (1+\theta_1)^t \): The convergence to the equilibrium output is controllable by policy. Contractionary money supply led to the Great Depression in 1929 within two years. Similarly, from (3.1), within as short as one year, \( t = \log p(x)/\theta_1 = \log(0.5)/\log(1+10.30) < 1 \), in the period 2001-2002 in Taiwan, declines in money growth led to the zero inflation rate and caused the rapid decrease of output growth towards 2\%. The large share of government spending and budget deficit is a partial cause.

4. Concluding Remarks

Using innovative dynamic quadratic regression, this paper proves the existence of the convergence probability and the equilibrium output, which is supported by the optimum in money supply and the share of budget deficit in GDP. The new econometric process proposed is, first, compute the equilibrium and optimal policy; second, reduce the heteroschedasticity; and third, estimate the convergence probability.

The maximum equilibrium is most acceptable to all participants and stable against deviations of any of \( n \) governments from the monetary union. We prove the positive cooperative equilibrium, while the Nash non-cooperative equilibrium [8] is biased downward, decided by the vote weights under politicization, and can not provide a solution acceptable to all government. The optimal policy oscillates over time and insulates noises. The super-performing government pays the incentive and education subsidies, while the under-performing government pays the taxes and penalties. The social welfare is Parato-improving for all horizons. The optimal policy, such as the share of government spending, income tax rates, and the share of real money balances in output, are individually rational and feasible while the budgets are closely balanced.

When output declines or output growth falls below the maximum equilibrium, the Keynesian expansionary policy is effective. Beyond the equilibrium, we accept the Ricardian (monetarist) equality constraint and Non-Ricardian (fiscalist) equilibrium as well as the hypothesis [3] in which the government spending crowds out private consumption. Some uncertainty of inflation can increase precautionary saving and investment. Too high inflation uncertainty has negative impacts upon employment and increases consumption more than saving and investment.

The state variables include the output, consumption, and unemployment.
rates. The optimum equilibrium satisfies the maximum principle; we minimize the error between the continuous solution and the discrete steady-state solution, which is stabilized by the optimal constrained policies. Beyond the optimum, the extreme policy is not optimal and stable. This paper provides hypothesis testing and proves the optimal budget deficit can lead to the unique optimum equilibrium of unemployment rates. Our optimum equilibrium is a turning point.

References


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