Properties of the Integral Equation Arising in the Valuation of American Options

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Abstract

Unlike European options, American options can be exercised at any time before the expiration date. This fact makes it difficult to analyze the price and the optimal exercise boundary of an American option. The optimal exercise boundary of an American option is implicitly defined by a nonlinear integral equation. This article studies the properties of the integral equation arising in the valuation of American options. Based on the properties of the integral equation, this article also presents a simple upper bound for the optimal exercise boundary of the American put.

Keywords: American option; Integral equation; Optimal exercise boundary

1. Introduction

Black and Scholes (1973) and Merton (1973) made a major breakthrough in the pricing of options. Their work has had a huge influence on financial management and financial engineering. In 1997, its importance was highlighted when Merton and Scholes were awarded the Nobel prize for economics, Fisher Black having passed away in 1995.

Ever since Black and Scholes (1973) derived a closed-form formula for European options, many researchers have been tried to obtain analytical results for American options. Unlike European options, American options can be exercised at any time before the expiration date. This fact makes it difficult to analyze the American option. So far a closed-form formula for the price of American options has not been found except for some special cases.

For this reason, many efforts have been concentrated on developing analytic approximation methods and numerical methods for the valuation of American options. Quadratic approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987), compound option approach of Geske and Johnson (1984) and Bunch and Johnson (1992), LUBA (lower and upper bound approximation) of Broadie and Detemple (1996), and the approximation method of Ju and Zhong (1999) are examples of analytic approximation methods.

For numerical methods, there are binomial method, finite difference method, Monte Carlo simulation method and integral representation method. The binomial method of Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979) and the finite difference method of Brennan and Schwartz (1977, 1978) are the earliest numerical methods and are still widely used. A comparison of these classical numerical methods can be found in Geske and Shastri (1985).


Integral representation method is based on the “integral representation” formula for the price of an American option which is independently derived by Kim (1990), Jacka (1991) and Carr, Jarrow, and Myneni (1992). In this integral representation formula, the American option price is the sum of otherwise identical European option price

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For example, Merton (1973) has shown that, if the underlying asset does not pay dividends, American calls will not be exercised early and will have the same value as equivalent European calls. Roll (1977), Geske (1979), and Whaley (1981) have derived a closed-form solution for the price of American call options with discrete dividends.

The integral representation formula requires the determination of the optimal exercise boundary for its implementation. To determine the optimal exercise boundary, we can apply “value-matching” or “high-contact” condition to generate nonlinear integral equations. However, it is not possible to obtain the optimal exercise boundary in explicit form. So, analytic and numerical properties of the optimal exercise boundary have been investigated by Kim and Byun (1994), Jacka (1991), and Peskir (2005). Kim, Byun, and Lim (2004) studies the numerical properties of the optimal exercise boundary in the case of deterministic volatility function. Asymptotic behavior of the optimal exercise boundary near the expiration time has also been investigated extensively in Kuske and Keller (1998), Bunch and Johnson (2000), Knessl (2001), Evans, Kuske, and Keller (2002) among others.

The main purpose of this article is to present the properties of the integral equation which must be satisfied by the optimal exercise boundary of an American option. These properties enable us to develop a simple upper bound for the optimal exercise boundary of the American put.

The article is organized as follows. Section 2 formalizes the problem of valuing American options. Section 3 studies the properties of the integral equation arising in the valuation of American options. Section 4 develops an upper bound for the optimal exercise boundary. The conclusions appear in the last section.

2. Integral Equation for the Optimal Exercise Boundary

Consider an American put option with exercise price of \( K \) that expires at time \( T \).

Assume that markets are perfect, trading occurs continuously, the rate of interest \( r \) is constant and the price of the underlying asset \( S(t) \) follows a lognormal diffusion process with volatility \( \sigma \) and expected return \( \mu \):

\[
    dS(t) = \mu S(t) dt + \sigma S(t) dW(t)
\]

where \( W(t) \) is a standard Brownian motion process. We also assume that the underlying asset pays continuous proportional dividends at a rate of \( \delta \).

Define the time to maturity, \( \tau = T - t \). Denote the American put option price by \( P(S, \tau) \) and the optimal exercise boundary by \( B^*_t \) as a function of the time to maturity. It is well known that they are the solution of a free boundary problem.

\[
    \frac{\sigma^2 S^2}{2} \frac{\partial^2 P}{\partial S^2} + (r - \delta) S \frac{\partial P}{\partial S} - \frac{\partial P}{\partial \tau} - rP = 0 \quad (2)
\]

subject to the following boundary conditions:

\[
    P(S,0) = \max [0, K - S] \quad (3)
\]

\[
    \lim_{\tau \to 0} P(S, \tau) = 0 \quad (4)
\]

\[
    \lim_{\tau \to 0} P(S, \tau) = K - B^*_t \quad (5)
\]

\[
    \lim_{\tau \to 0} \frac{\partial P(S, \tau)}{\partial S} = -1 \quad (6)
\]

Equation (5) is the “value-matching” condition and Equation (6) is the “high-contact” condition. These conditions ensure the optimality of the exercise boundary \( B^*_t \). Kim (1990), Jacka (1991) and Carr, Jarrow, and Myneni (1992) have obtained the following “integral representation” formula for an American put:

\[
    P(S, \tau) = p(S, \tau) + \frac{\gamma}{\sqrt{2\pi}} \int_0^\infty \left[ Ke^{-\delta(\tau-s)} N(-d_2(S, B^*_t, \tau - s)) \right]
\]

\[
    - \delta S e^{-\delta(\tau-s)} \int_0^\infty \left[ d_1(S, B^*_t, \tau - s) \right] ds \quad (7)
\]

where

\[
    d_1(x, y, \tau - s) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)(\tau - s)}{\sigma \sqrt{\tau - s}}
\]

\[
    d_2(x, y, \tau - s) = d_1(x, y, \tau - s) - \sigma \sqrt{\tau - s}
\]

Here \( N(*) \) is the standard cumulative normal distribution function and \( p(S, \tau) \) is the price of the corresponding Black and Scholes (1973) European put option formula.

The integral representation formula (7) for the American option requires the determination of the optimal exercise boundary for its implementation. To determine the optimal exercise boundary, we can apply “value-matching” condition (5) or “high-contact” condition (6) to generate nonlinear integral equations. By imposing the “value-matching” condition (5), the following integral equation is obtained:

\[
    K - B^*_t = p(B^*_t, \tau) + \frac{\gamma}{\sqrt{2\pi}} \int_0^\infty \left[ Ke^{-\delta(\tau-s)} N(-d_2(B^*_t, B^*_t, \tau - s)) \right]
\]

\[
    - \delta B^*_t e^{-\delta(\tau-s)} \int_0^\infty \left[ d_1(B^*_t, B^*_t, \tau - s) \right] ds \quad (8)
\]
Equation (8) is a nonlinear Volterra integral equation of the second kind [see Linz (1985)]. Little, Pant, and Hou (2000) derives an alternative integral equation for the optimal exercise boundary of an American put option. The following section investigates the properties of the integral equation (8).

3. Properties of the Integral Equation

As studied in Carr, Jarrow, and Myneni (1992) and Broadie and Detemple (1996), consider the class of contracts consisting of a European put option and a sure flow of payments that are paid at the rate

\[ rKN(-d_2(S,B_s,\tau-s)) - \delta e^{-(\delta \tau)(s)} N(-d_1(S,B_s,\tau-s)) \]

(9)

for \( s \in [0, \tau] \), where \( B_s \) is a nonnegative piecewise continuous function of time to maturity. Each member of the class of contracts is parameterized by \( B \). The value of the contract is

\[ V(S,B) = p(S,\tau) + \int_0^\tau \Phi(S,B_s,s)ds \]

(10)

where

\[ \Phi(S,B_s,s) = rKe^{-r(t-s)} N(-d_2(S,B_s,\tau-s)) \]

\[ - \delta S e^{-\delta (t-s)} N(-d_1(S,B_s,\tau-s)) \]

and \( p(S,\tau) \) is the Black and Scholes (1973) European put option pricing formula. European and American put options are special contracts with \( B_s = 0 \) and \( B_s = B_s^* \), respectively, where \( B_s^* \) is the optimal exercise boundary of the American put. The optimal exercise boundary solves the following integral equation:

\[ K - B_s^* = p(B_s^*,\tau) + \int_0^\tau \Phi(B_s^*,B_s^*,s)ds \]

(11)

For a completeness, we define \( \Phi(S,B_s,s) \) when \( S = 0 \) as

\[ \Phi(0,B_s,s) = \lim_{S \to 0} \Phi(S,B_s,s) = rKe^{-r(t-s)} \]

(12)

Now we consider the maximum stock price \( S^* \) which satisfy the following equation for an arbitrary continuous function \( B \).

\[ S^* = \max \left\{ S : K - S = p(S,\tau) + \int_0^\tau \Phi(S,B_s,s)ds \right\} \]

(13)

Note that \( S^* \) is a functional of the function \( B \). The solution of Equation (13) \( S^* \) has the following properties.

**Proposition 1.** Let \( S \) denote the solution of Equation (13), then

(i) For any positive function \( B_s^* > 0 \) for all \( s \in [0, \tau] \), \( S = 0 \) is in the set in Equation (13), thus we have \( S^* \geq 0 \).

(ii) When \( B_s \to \infty \) for all \( s \in [0, \tau] \), we have \( S^* = 0 \).

(iii) When \( B_s^* \geq (r/\delta)K \) for all \( s \in [0, \tau] \), we have \( S^* = 0 \).

Property (i) is the reason why we used “max” in Equation (13). Note that property (ii) is a special case of property (iii). We are most interested in the case where \( B_s \leq (r/\delta)K \). The following theorem touches this case.

**Theorem 1.** Let \( S^*_1 \) and \( S^*_2 \) denote the solution of Equation (13) with \( B_s = B_s^* \) and \( B_s = B_s^*(1) \), respectively.

If \( 0 < B_s^* \leq B_s^*(2) \leq (r/\delta)K \) for all \( s \in [0, \tau] \), then \( S^*_1 \geq S^*_2 \).

Theorem 1 implies that if we have a function \( B_s \) for all \( s \in [0, \tau] \), then the solution of Equation (13) with \( B_s^* \) gives an upper bound for the optimal exercise boundary.

\[ S^* (B_s) = S^* (B_s^*(1)) \geq S^* (B_s^*(2)) = B_s^* \]

In the case of constant functions \( B_s = B \) for \( B \in (0, (r/\delta)K] \). Theorem 1 implies the following proposition.

**Proposition 2.** Let \( B_s^* \) denote the optimal exercise boundary for the American put option. Then,

(i) \( S^* (B_s = B) \geq B_s^* \) for \( B \in (0, B_s^*] \)

(ii) \( S^* (B_s = B) \leq B_s^* \) for \( B \in [B_s^*, (r/\delta)K] \)

(iii) \( S^* (B_s = B) \) is a decreasing function of \( B \) for \( B \in (0, (r/\delta)K] \)

Properties (i) and (ii) are from the fact that the optimal exercise boundary of an American put is a decreasing function of the time to maturity. Property (i) says that the solution of Equation (13) \( S^* \) for \( B \in (0, B_s^*] \) lies uniformly above the optimal exercise boundary \( B_s^* \).

Property (ii) shows that the solution of Equation (13) \( S^* \) for \( B \in [B_s^*, (r/\delta)K] \) lies uniformly below the optimal exercise boundary \( B_s^* \). From property (iii) the most tight lower and upper bounds for the optimal exercise boundary are \( S^* (B_s = B_s^*) \leq S^* (B_s = B_s^*) \).

4. An Upper Bound for the Optimal Exercise Boundary

This section gives an upper bound for the optimal exercise boundary of an American put. Consider the solution \( B^* \) of the equation \( S^* (B_s = B_s^*) = B^* \). Then \( B^* \) is the solution to the equation,
The following is the main result of this article. This article reviews the literature for valuing American options and studies the properties of the optimal exercise problem of valuing American options has a large literature in financial engineering and financial management field. The properties of \( B^* \) are summarized in the following theorem. We denote \( B^a \) as a function of the time to maturity \( \tau \). The following is the main result of this article.

**Theorem 2.** Let \( B^* \) denote the optimal exercise boundary for the American put option. Let \( B^a \) denote the solution to Equation (14). Then,

(i) \( B^a \geq B^* \)

(ii) \( \lim_{\tau \to 0} B^* = B^a_0 = B^*_0 \)

(iii) \( \lim_{\tau \to \infty} B^a = B^*_\infty = B^*_\infty \)

Theorem 2 part (i) says that the solution of Equation (14) \( B^a \) lies uniformly above the optimal exercise boundary \( B^* \). Parts (ii) and (iii) show that \( B^* \to B^a \) in two limiting cases. It should be noted that since \( B^a \) lies uniformly above the optimal exercise boundary \( B^* \), \( S^* (B_S = B^*_S) \leq B^a \leq B^*_S \) gives the most tight lower and upper bounds for the optimal exercise boundary and the value of the American put is bounded above by

\[
p (S, \tau) + \int_0^\tau \Phi (\Phi (S, B, \tau), s) ds \tag{14}\]

5. Conclusion

One of the exciting developments in financial markets over the last 30 years has been the growth of derivatives markets. This explosive growth in the use of derivatives has needed their efficient and accurate valuation. The problem of valuing American options has a large literature in financial engineering and financial management field. This article reviews the literature for valuing American options and studies the properties of the optimal exercise boundary.

The optimal exercise boundary of the American option is implicitly defined by a nonlinear integral equation. This article studies the properties of the integral equation and presents a simple upper bound for the optimal exercise boundary of American put options.

References


Appendix

Proof of Proposition 1.

(i) Since \( \lim_{S \to 0} \Phi(S, B_s, s) = rKe^{-r(\tau-s)} \), the integral in Equation (10) can be evaluated in closed form:

\[
\lim_{S \to 0} \left[ p(S, \tau) + \int_0^\tau \Phi(S, B_s, s) \, ds \right] = K
\]

Thus, \( S = 0 \) satisfies the equality in Equation (13) and we have \( S^* \geq 0 \).

(ii) Note that \( \Phi(S, B_s, s) \) satisfies

\[
\lim_{B_s \to \infty} \Phi(S, B_s, s) = rKe^{-r(\tau-s)} - \delta e^{-\delta(\tau-s)}
\]

Hence, when \( B_s \to \infty \) for all \( s \in [0, \tau] \), we have

\[
p(S, \tau) + \int_0^\tau \Phi(S, B_s, s) \, ds = K - S + c(S, \tau)
\]

where \( c(S, \tau) \) is the Black and Scholes (1973) European call option pricing formula. Therefore \( S = 0 \).

(iii) For each \( s \in [0, \tau] \), consider \( \Phi(S, B_s, s) : \mathbb{R}^+ \to \mathbb{R} \) defined in Equation (10) as a function of \( B_s \). Then the partial derivative of \( \Phi(S, B_s, s) \) with respect to \( B_s \) is

\[
\frac{\partial \Phi(S, B_s, s)}{\partial B_s} = rKe^{-r(\tau-s)} \frac{(-d_B(S, B_s, s))}{B_s \sigma \sqrt{t-s}}
\]

\[
- \delta e^{-\delta(\tau-s)} \frac{(-d_B(S, B_s, s))}{B_s \sigma \sqrt{t-s}}
\]

\[
= (rK - \delta B_s) e^{-r(\tau-s)} \frac{(-d_B(S, B_s, s))}{B_s \sigma \sqrt{t-s}}
\]

where we used the following identity

\[
B_t e^{-r(t-s)}(-d^2(S, B_t, \tau-s)) = S e^{K(\tau-s)}(-d^1(S, B_t, \tau-s))
\]

and \( n(*) \) denotes the standard normal density function. Thus, it can be easily checked that \( \Phi(S, B_s, s) \) is maximum at \( B_s = (r/\delta)K \), strictly increasing for \( B_s \in (0, (r/\delta)K) \) and strictly decreasing for \( B_s \in ((r/\delta)K, \infty) \).

Thus, if we consider the function \( B_s \) such that \( (r/\delta)K \leq B_s < \infty \) for all \( s \in [0, \tau] \), then we have

\[
p(S, \tau) + \int_0^\tau \Phi(S, B_s, s) \, ds \geq p(S, \tau) + \int_0^\tau \Phi(S, \infty, s) \, ds = K - S + c(S, \tau)
\]

Therefore, from part (i), it follows that \( S^* = 0 \).

Proof of Theorem 1.

As shown in Proposition 1 part (iii), \( \Phi(S, B_s, s) \) is a strictly increasing function of \( B_s \), where \( B_s \in (0, (r/\delta)K) \). Thus, if we consider two functions \( B_s^{(1)} \) and \( B_s^{(2)} \) such that \( 0 < B_s^{(1)} < B_s^{(2)} \) for all \( s \in [0, \tau] \), then we have

\[
\int_0^\tau \Phi(S, B_s^{(1)}, s) \, ds < \int_0^\tau \Phi(S, B_s^{(2)}, s) \, ds < \int_0^\tau \Phi(S, (r/\delta)K, s) \, ds
\]

for all \( S \geq 0 \). Therefore, from Proposition 1 part (i), it follows that \( S^{*1} \geq S^{*2} \geq 0 \).

Proof of Proposition 2.

(i) Since the optimal exercise boundary is a decreasing function of the time to maturity, \( B_s^{*} \geq B^{*} \) for \( s < \tau \). Theorem 1 implies that

\[
S^{*}(B_s = B_s) \geq S^{*}(B_s = B^{*}) = B^{*}
\]

for \( B \in [0, B_s^{*}] \).

(ii) Since \( B_s^{*} < B^{*} \) for \( s > 0 \), Theorem 1 implies that

\[
S^{*}(B_s = B_s) \leq S^{*}(B_s = B^{*}) = B^{*}
\]

for \( B \in [B_s^{*}, (r/\delta)K] \).

(iii) Let \( 0 \leq B_s \leq B^{*} \leq (r/\delta)K \). Then Proposition 1 says that

\[
S^{*}(B_s = B) \geq S^{*}(B_s = B_s) = B_s
\]

Therefore \( S^{*}(B_s = B_s) \) is a decreasing function of \( B \) for \( B \in [0, (r/\delta)K] \).

Proof of Theorem 2.

(i) Since \( B_s^{*} = B_s^{*} \) and \( S^{*}(B_s = B_s) = B_s^{*} \) is a decreasing function of \( B \), we have \( B_s^{*} < B_s^{*} \).

(ii) We first show that \( B^{*} \leq B^{*} \). Since \( S^{*}(B_s = B^{*}) \leq B^{*} \) and \( S^{*}(B_s = B^{*}) \) is a decreasing function of \( B \), we have \( S^{*}(B_s = B^{*}) \leq S^{*}(B_s = B^{*}) \). Thus \( B_s^{*} \leq B^{*} \).

We have \( B^{*} \to B^{*} \) as \( \tau \to \infty \).

(iii)Since \( B_s^{*} \leq B^{*} \leq S^{*}(B_s = B^{*}) \) and \( S^{*}(B_s = B^{*}) \to S^{*} \) as \( \tau \to \infty \), we have \( B^{*} \to B^{*} \) as \( \tau \to \infty \).