Determining the Optimal Delivery Schedule of Spare Units via NPV Approach

Tadashi Dohi\textsuperscript{a,}\textsuperscript{*}, Takashi Danjou\textsuperscript{a} and Naoto Kaio\textsuperscript{b}

\textsuperscript{a}Department of Information Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan.
\textsuperscript{b}Department of Economic Informatics, Hiroshima Shudo University, 1-1-1 Ozukahigashi, Asaminami-ku, Hiroshima 731-3195, Japan.

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Abstract

In this article, we consider a stochastic profit model with discounting for a simple delivery scheduling problem of spare units and maximize the expected total discounted profit over an infinite time planning horizon. Then the familiar net present value (NPV) approach is applied to formulate the present value of expected total profit for an infinite time planning span. In a fashion similar to Dohi \textit{et al.} (2000), we further propose a statistical estimation procedure for the optimal delivery schedule, based on the non-parametric method. More precisely, the concept of total time on test statistics is utilized for the development of a non-parametric estimation method. In a simulation study, we investigate numerically asymptotic properties of the resulting estimator of the optimal delivery schedule. It can be concluded that the developed method here can provide nice convergence properties to the real optimal solution and be useful for actual spare parts management.

\textit{Keywords:} Spare part inventory management; Delivery schedule; Profit model; Discounting; Non-parametric estimation

1. Introduction

Since the seminal contributions by Barlow and Proschan (1996) and Barlow (1998), a number of optimal maintenance models under uncertainty have been developed in the literature. In the usual maintenance models, the corrective replacement after system failure occurs is assumed to be performed immediately without any time delay. However, in many practical situations such as manufacturing processes, the spare units or parts are not always available on hand, and may be delivered periodically. Wiggins (1967), Allen and D'Esopo (1968), Nakagawa and Osaki (1974), Nakagawa (1976) and Dohi \textit{et al.} (1998a, 1998c, 2003b, 2004) treat the order-replacement problems for a non-repairable unit with lead times. In particular, Nakagawa (1976) discusses a simple but interesting spare part delivery scheduling problem under the long-run profit per unit time in the steady state. More specifically, it is assumed that the spare part can be delivered at a pre-specified time, and that the number of spare part in inventory is at most one. These assumptions do not meet the JIT (Just-in-Time) production philosophy, but can be validated for the manufacturing processes with one machine in subcontractor factories. For a comprehensive bibliography in the spare part delivery scheduling problems, see Dohi \textit{et al.} (1998b, 2005).

It should be noted in Nakagawa (1976), however, that the optimal delivery schedule which maximizes the long-run profit per unit time in the steady state is derived analytically, provided that the lifetime distribution of the part is known completely. In other words, since the lifetime of the manufacturing part strongly depends on the frequency for use and the production environment in the factory, it is not easy to collect a sufficient number of lifetime data to characterize the lifetime distribution function in the operational phase. Hence, any statistical estimation method to determine the optimal delivery schedule from the observation data will be useful to realize an adaptive maintenance control scheme in the manufacturing process. Dohi \textit{et al.} (2000) develop a non-parametric estimation method of the optimal spare delivery schedule, based on the scaled total time on test (TTT) concept (Barlow and Campo, 1975). First, Bergman (1979), Bergman and Klefsjo (1984) and Barlow and Davis (1977) develop statistical non-parametric methods to determine the optimal age replacement time and the optimal periodic replacement time. Dohi \textit{et al.} (1998a) propose non-parametric preventive maintenance optimization models under earning rate criteria. Bergman and Klefsjo (1983) develop the modified TTT-transform/statistics, by taking account of time discounting, and apply them to analyze an age replacement with discounting. Dohi \textit{et al.} (2003a) also propose a new graphical method to estimate the optimal repair-time limit under the discounted cost criterion.

\textsuperscript{*} Email: dohi@rel.hiroshima-u.ac.jp
In this article, we consider the same delivery scheduling problem of spare units as Nakagawa (1976) under the discounted profit criterion. In Section 2, we review the result of Nakagawa (1976) and give some mathematical results on the optimal spare part delivery schedule maximizing the long-run profit per unit time in the steady state. Section 3 concerns the maximization problem of the expected total discounted profit. We formulate the expected total discounted profit over an infinite time horizon, based on the familiar net present value (NPV) approach, and derive analytically the optimal spare part delivery schedule which maximizes it. In Section 4, we propose the graphical methods to determine the optimal delivery schedule in the two-dimensional plane, which consist of both the cumulative operation time and the scaled TTT transform of lifetime distribution function. Next, suppose that an ordered complete sample data from the unknown lifetime distribution function is given. Following the idea by Bergman and Klefsjo (1983) with the modified scaled TTT statistics, non-parametric estimation methods are developed to determine the optimal delivery schedule, from the complete lifetime data. Numerical examples are devoted in Section 5 to examine asymptotic properties of the resulting estimator of the optimal delivery schedule. Finally, the article is concluded with some remarks in Section 6.

2. Related Work

2.1 Notation and Assumptions

\[ F(t), f(t), r(t), (> 0): \text{cdf, pdf, failure rate and mean of continuous lifetime of the part} \]

\[ \psi(\cdot) = 1 - \psi(\cdot) \text{ in general} \]

\[ t_0 (\geq 0): \text{delivery time or the delivery schedule (decision variable)} \]

\[ q (> 0): \text{profit per unit operating time in manufacturing} \]

\[ c_1 (> 0): \text{system down cost per unit time} \]

\[ c_2 (> 0): \text{inventory holding cost per unit time for spare part} \]

2.2 Spare Part Delivery Problem

Following Nakagawa (1976), we describe a stochastic delivery scheduling problem of spare parts. Consider a manufacturing machine with one repairable unit. Suppose that the manufacturing machine is operated in continuous time and starts working at time \( t = 0 \). Let \( F(t) \) be the lifetime distribution function of the part, with density \( f(t) (> 0) \), failure rate \( r(t) = f(t)/F(t) \) and finite mean \( \mu (> 0) \). Without any loss of generality, the distribution function \( F(t) \) is absolutely continuous and has its inverse \( F^{-1}(\cdot) \). The spare part is ordered at time \( t_0 \) periodically. If the manufacturing machine fails up to time \( t_0 \), then the system down occurs until the delivery of spare part.

Hence, the failed part is replaced by new one at time \( t_0 \).

On the other hand, if the manufacturing machine does not fail until time \( t_0 \), then the spare part may be put in inventory and the failed part after \( t = t_0 \) is replaced by the spare one. Throughout this article, the replacement time is sufficiently small and can be negligible. Since the spare part is rather expensive and the inventory holding cost is relatively large, it is assumed that the number of spare part in inventory is at most one. Figure 1 illustrates the configuration of the present spare part delivery scheduling problem.

2.3 Optimal Spare Part Delivery Scheduling

Define the time interval from the beginning of the ordering of a spare part to the next one as one cycle. Since each of these points is a renewal point, the stochastic process under consideration can be regarded as a renewal reward process (Ross, 1970). The expected total profit for one cycle is

\[
EC(t_0) = q\mu - c_1 \int_0^{t_0} (t_0 - t) dF(t) - c_2 \int_0^{t_0} (t - t_0) dF(t)
= q\mu - c_1 t_0 - c_2 \mu + (c_1 + c_2) \int_0^{t_0} F(t) dt.
\]

Also, the mean time length of one cycle is given by

\[
ET(t_0) = \int_0^{t_0} t dF(t) + \int_0^{t_0} dF(t) = \mu + t_0 - \int_0^{t_0} F(t) dt.
\]

From the familiar renewal reward argument (Ross, 1970), the long-run profit per unit time in the steady state \( P(t_0) \) is given by

\[
P(t_0) = EC(t_0)/ET(t_0).
\]

Then the problem is to find the optimal delivery schedule \( t_0 \) such as

\[
t_0^* = \max \left\{ t_0 \geq 0 \mid \text{max P}(t_0) \right\}.
\]
where \( t_0 = \sup\{t_0 \geq 0 \mid P(t_0) \geq 0\} = \sup\{t_0 \geq 0 \mid EC(t_0) \geq 0\} \)
to guarantee the positive profit level.

**Lemma 2.1:** Suppose that the density function \( f(t) \) is absolutely continuous and strictly positive, and that \( q > c_2 \). Then, there exists a unique upper limit \( t_0 \).

**Proof:** From Eq.(1), it is found that \((q - c_2) \geq V(t_0) \)
if \( EC(t_0) \geq 0 \), where
\[
V(t_0) = c_1 \int_0^{t_0} F(t)dt - c_2 \int_0^{t_0} \bar{F}(t)dt.
\]
(5)

Since \( dV(t_0) / dt_0 = (c_1 + c_2) f(t_0) > 0 \),
\[
\lim_{t_0 \to 0} dV(t_0) / dt_0 = -c_2 < 0 \quad \text{and} \quad \lim_{t_0 \to \infty} dV(t_0) / dt_0 = c_1 > 0\),
the function \( dV(t_0) / dt_0 \) crosses the zero level at \( t_0 = F^{-1}(c_2 / (c_1 + c_2))\). Hence, the function \( V(t_0) \)
is monotonically decreasing in \( t_0 \in [0, F^{-1}(c_2 / (c_1 + c_2))] \) and is monotonically increasing in \( t_0 \in (F^{-1}(c_2 / (c_1 + c_2)), t_0] \), where \( V(t_0) = (q - c_2) > 0 \). If the density function \( f(t) \) is absolutely continuous, there exists such an upper limit \( t_0 \) uniquely.

Define the numerator of the derivative of \( P(t_0) \) with respect to \( t_0 \), as \( q(t_0) \), i.e.
\[
q(t_0) = \left[ (c_1 + c_2) \bar{F}(t_0) - c_1 \right] \alpha T(t_0) - F(t_0) EC(t_0).
\]
(6)

The following result is derived by Nakagawa (1976), although it was not described in a systematic way.

**Theorem 2.2:** (i) If \( q(t_0) < 0 \), there exists a finite and unique optimal delivery schedule \( t_0^* (0 < t_0^* < t_0) \) satisfying \( q(t_0^*) = 0 \) and the maximum long-run profit per unit time in the steady state is given by
\[
P(t_0^*) = \frac{(c_1 + c_2) \bar{F}(t_0^*) - c_1}{F(t_0^*)}.
\]
(7)

(ii) If \( q(t_0) \geq 0 \), then the function \( P(t_0) \) is monotonically increasing and the optimal delivery schedule is given by \( t_0^* = t_0 \).

**Proof:** Differentiating \( q(t_0) \) and setting it equal to zero yield
\[
\frac{d}{dt_0} q(t_0) = \left[ (c_1 + c_2) \alpha T(t_0) + EC(t_0) \right] f(t_0) < 0
\]
(8)
from \( t_0 \in [0, t_0] \). This implies that the function \( P(t_0) \) is strictly concave in \( t_0 \in [0, t_0] \). Since \( q(0) = c_2 > 0 \), the proof is completed. Q.E.D.

### 3. NPV Approach

**3.1 Formulation of Expected Total Discounted Profit**

Next, let us define the discount factor \( \alpha (\alpha > 0) \) to represent the net present value (NPV) of the total expected profit over an infinite time horizon. The NPV of the system down cost for one cycle is
\[
\int_0^t \int_0^t c e^{-\alpha t} dF(t) = \frac{c_1 (1 - e^{-\alpha t_0})}{\alpha} - c_1 \int_0^t e^{-\alpha t} F(t)dt.
\]
(9)

The NPV of the inventory holding cost is given by
\[
\int_0^t \int_0^t c e^{-\alpha t} dF(t) = c_2 \int_0^t e^{-\alpha t} F(t)dt - c_1 \int_0^t e^{-\alpha t} F(t)dt.
\]
(10)

Hence the expected total discounted profit for one cycle is given by
\[
EC_\alpha(t_0) = q \int_0^t \frac{1 - e^{-\alpha t}}{\alpha} dF(t) = \frac{c_1 (1 - e^{-\alpha t_0})}{\alpha} - c_1 \int_0^t e^{-\alpha t} F(t)dt + (c_1 + c_2) \int_0^t e^{-\alpha t} F(t)dt.
\]
(11)

On the other hand, the expected unit cost can be discounted after one cycle as follows:
\[
\delta_\alpha(t_0) = \int_0^t e^{-\alpha t_0} dF(t) + \int_0^t e^{-\alpha t_0} dF(t) = e^{-\alpha t_0} - \alpha \int_0^t e^{-\alpha t} F(t)dt.
\]
(12)

Finally, we derive the expected total discounted profit over an infinite time horizon as
\[
P_\alpha(t_0) = EC_\alpha(t_0) + \delta_\alpha(t_0) EC_\alpha(t_0) + \delta_\alpha^2(t_0) EC_\alpha(t_0) + \cdots
\]
\[
= \sum_{n=0} \delta_\alpha^n(t_0) EC_\alpha(t_0) = EC_\alpha(t_0) / \delta_\alpha(t_0).
\]
(13)

**Theorem 3.1:**
\[
\lim_{\alpha \to 0} \alpha P_\alpha(t_0) = P(t_0).
\]
(14)

The proof is due to the direct application of de L’Hospital’s theorem.

**Theorem 3.2:** Suppose that the density function \( f(t) \) is absolutely continuous and strictly positive, and that \( q > c_2 \). Define \( t_{1/\alpha} = \sup\{t_0 \geq 0 \mid P(t_0) \geq 0\} = \sup\{t_0 \geq 0 \mid EC_\alpha(t_0) \geq 0\} \). If \( EC_\alpha(\infty) > (q - c_2) \int_0^\infty \exp(-\alpha t) F(t)dt \) then there exists a unique \( t_{1/\alpha} \) satisfying
\[
EC_\alpha(t_{1/\alpha}) = (q - c_2) \int_0^\infty e^{-\alpha t} F(t)dt.
\]
(15)
otherwise \( t_{1/\alpha} \to \infty \).
3.2 Optimal Spare Part Delivery Scheduling

Define the numerator of the derivative of \( P_d(t_0) \) with respect to \( t_0 \), divided by \( \exp(-\alpha t_0) \), as \( q_d(t_0) \),

\[
q_d(t_0) = \{(c_1 + c_2)\bar{F}(t_0) - c_1 \mathcal{P}_d(t_0) - \alpha F(t_0)\} \mathcal{E}_a(t_0),
\]

where obviously

\[
\lim_{\alpha \to 0 -} q_d(t_0) = q(t_0).
\]

**Theorem 3.3:** (i) If \( q_d(t_{UA}) < 0 \), there exists a finite and unique optimal delivery schedule \( t_0^* \) \((0 < t_0^* < t_{UA})\) satisfying \( q_d(t_0^*) = 0 \) and the maximum expected total discounted profit over an infinite time horizon is given by

\[
P_a(t_0^*) = \frac{(c_1 + c_2)\bar{F}(t_0^*) - c_1 \mathcal{P}_d(t_0^*)}{\alpha F(t_0^*)}.
\]

(ii) If \( q_d(t_{UA}) \geq 0 \), then the function \( P_a(t_0) \) is monotonically increasing and the optimal delivery schedule is given by \( t_0^* = t_{UA} \).

**Proof:** Differentiating \( q_d(t_0) \) and setting it equal to zero yield

\[
\frac{d}{dt_0} q_d(t_0) = -\left\{ (c_1 + c_2) \mathcal{P}_d(t_0) + \alpha \mathcal{E}_a(t_0) \right\} f(t_0) < 0
\]

from \( t_0 \in [0, t_{UA}] \). This implies that the function \( P_d(t_0) \) is strictly concave in \( t_0 \in [0, t_{UA}] \). Since \( q_d(0) = c_2 \mathcal{P}_d(0) > 0 \), the proof is completed.

Q.E.D.

In the following section, we consider statistical estimation problems of the optimal spare part delivery scheduling under the expected profit criteria.

4. Statistical Estimation Algorithms

4.1 Graphical Problems

To derive the optimal delivery time on the graph, we define the equilibrium distribution or equivalently the scaled total time on test transform of the lifetime distribution (Barlow and Compo, 1975) by

\[
\phi(t_0) = \int_0^\infty F(t)dt / \mu.
\]

Then, the long-run profit per unit time in the steady state is given by

\[
P(t_0) = -\left( c_1 + c_2 \right) + \frac{q + c_1 + c_2 t_0 / \mu}{1 + t_0 / \mu - \phi(t_0)}.
\]

The following result is similar to Theorem 2.2, but is useful to interpret the underlying optimization problem geometrically (Dohi et al., 1998b).

**Theorem 4.1:** Obtaining the optimal delivery schedule which maximizes the long-run profit per unit time in the steady state can be reduced to the following maximization problem:

\[
\max_{0 < t_0 < t_{UA}} \frac{\phi(t_0) + \xi}{t_0 + \eta},
\]

where

\[
\eta = (q + c_1) \mu / c_2
\]

and

\[
\xi = (q + c_1) / c_2 - 1.
\]

The proof can be made through a few algebraic manipulations (see Dohi et al. (2000)). From Theorem 4.1, it is found that the underlying delivery scheduling problem of spare units is reduced to obtain the point \( t_0^* \) so as to maximize the tangent slope from the point \( S(-\eta, -\xi) \) to
the curve \( \phi(t_0) \) on the plane \((x, y) = (t_0, \phi(t_0))\). This problem is quite different from the earlier ones considered in Dohi et al. (1998a, 2003), Bergman (1979) and Bergman and Klefsjo (1984), since the existing methods treat a somewhat different one on the plane \((x, y) = (F(t_0), \phi(t_0))\).

Next, consider the discounting problem. Following Bergman and Klefsjo (1983), we define the modified scaled total time on test transform of the lifetime distribution by

\[
\phi_a(t_0) = \int_0^{t_0} F_a(t)dt / \mu_a, \tag{29}
\]

\[
F_a(t) = 1 - e^{-at}F(t), \tag{30}
\]

\[
\mu_a = \int_0^\infty F_a(t)dt. \tag{31}
\]

Then, the expected total discounted profit over an infinite time horizon is rewritten as

\[
P_a(t_0) = \frac{\left( c_1 / \alpha \right) \left[ 1 - e^{-at_0} \right] + (c_1 + c_2) \mu_a \phi_a(t_0) + q \mu_a}{(1 - e^{-at_0}) - \alpha \mu_a \phi_a(t_0) + \alpha \mu_a}. \tag{32}
\]

**Theorem 4.2:** Obtaining the optimal delivery schedule which maximizes the expected total discounted profit over an infinite time horizon can be reduced to the following maximization problem:

\[
\max_{0 \leq t_0 \leq \infty} \frac{\phi_a(t_0) + \xi_a}{1 - e^{-at_0} + \eta_a}, \tag{33}
\]

where

\[
\eta_a = \frac{\alpha \mu_a + (c_1 + c_2)q \mu_a}{c_1 + c_2 + 1} \tag{34}
\]

and

\[
\xi_a = \frac{\alpha q - \alpha}{\alpha(c_1 + c_2 + 1)}. \tag{35}
\]

The proof is similar to that of Theorem 4.1, since the maximization problem \( \max_{0 \leq t_0 \leq \infty} P_a(t_0) \) can be reduced to Eq.(33). Hence, the optimal spare part schedule can be characterized by finding the point \( t_0^* \) so as to maximize the tangent slope from the point \( \hat{S}_a(-\eta_a - \xi_a) \) to the curve \( \phi_a(t_0) \) on the plane \((x, y) = (1 - \exp(-\alpha t_0), \phi_a(t_0))\).

4.2 Non-parametric Estimators

Suppose that an ordered complete sample \( 0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \) from the underlying lifetime distribution function \( F \), which is unknown, is available. The scaled TTT statistics based on this sample is defined by

\[
\phi_{a,i} = T_i / T_n, \quad i = 0, 1, 2, \ldots, n, \tag{36}
\]

where

\[
T_i = \sum_{j=1}^{i} \left( 1 - \frac{j-1}{n} \right) (x_j - x_{j-1}), \quad T_0 = 0. \tag{37}
\]

It is well known that \( \phi_{a,i} \) is an estimator of the scaled TTT transform \( \phi(t_0) \). That is, plotting the point \((x_i, \phi_{a,i}), i = 0, 1, 2, \ldots, n \) and connecting them by line segments yield a non-parametric estimator of the function \( \phi(t_0) \).

The following result is a direct application to Theorem 3.1.

**Theorem 4.3:** Suppose that an ordered complete sample \( 0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \) from the unknown lifetime distribution function is observed. Then, an estimator of the optimal delivery schedule \( t_0^* \) maximizing the long-run profit per unit time in the steady state is given by

\[
i_{0,n}^* = \left\{ x_i \mid \max_{0 \leq i \leq n} \frac{\phi_{a,i} + \xi_a}{x_i + \eta_a} \right\}, \tag{38}
\]

where

\[
\eta = (q + c_2) \hat{\mu}_a / c_2, \tag{39}
\]

\[
\hat{\mu}_a = \sum_{j=1}^{n} x_j / n, \tag{40}
\]

\[
n_i = \max\{k \geq 0 \mid \hat{\mu}_a(q - c_2) \geq V_{a,i} \}, \tag{41}
\]

and

\[
V_{a,i} = c_j \sum_{j=1}^{k} (j-1)(x_j - x_{j-1}) - c_j \sum_{j=1}^{k} (n - j + 1)(x_j - x_{j-1}). \tag{42}
\]

The proof is omitted for brevity. In fact, the estimator given in Theorem 4.3 is strongly consistent, i.e. \( i_{0,n}^* \to t_0^* \) as \( n \to \infty \), a.s. This result can be expected easily from Theorem 3.1. Also, the strongly consistent property for the scaled TTT statistics is due to Barlow and Campo (1975).

Next, following Bergman and Klefsjo (1983), let us define the modified scaled TTT statistics based on this sample by

\[
\phi_{a,i} = T_{ia} / T_{na}, \quad i = 0, 1, 2, \ldots, n, \tag{43}
\]

where

\[
T_{ia} = \sum_{j=1}^{i} \left( 1 - \frac{j-1}{n} \right) (x_j - x_{j-1})e^{-xs_j}, \quad T_{na} = 0. \tag{44}
\]

Plotting the point \((1 - e^{-\alpha s}, \phi_{a,i}), i = 0, 1, 2, \ldots, n \) and connecting them by line segments yield the modified curve for the discounting problem.
Theorem 4.4: Suppose that an ordered complete sample \( 0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \) from the unknown lifetime distribution function is observed. Then, an estimator of the optimal delivery schedule \( t_0^* \) maximizing the expected total discounted profit over an infinite time horizon is given by

\[
\hat{t}_{0,n,a} = \left\{ x_i \mid \max_{0 \leq m \leq n} \phi_{n,a} + \hat{\eta}_a \right\},
\]

(45)

where

\[
\hat{\eta}_a = \frac{(\alpha + c_1 + c_2)q\hat{\mu}_{n,a}}{c_1 + c_2 + 1},
\]

(46)

\[
n_{n,a} = \max\{k \geq 0 \mid \hat{\mu}_{n,a} (q - c_2) \geq V_{l,n,a}\}
\]

(47)

\[
\hat{\mu}_{n,a} = \frac{\sum_{i=1}^{\hat{\gamma}} (n - j + 1) (x_j - x_{j-1}) e^{-s_j}}{n}.
\]

(48)

and

\[
V_{l,n,a} = (c_1 / \alpha)[1 - e^{\alpha x_0}] + (c_1 + c_2) \hat{\mu}_{n,a} \phi_{l,n,a} + q \hat{\mu}_{n,a}.
\]

(49)

The result above can be shown easily by replacing \((1 - e^{-\alpha x_0}), \phi_{a}(t_0)\) by \((1 - e^{-\alpha x_0}), \phi_{l,n,a}\) in Theorem 4.2.

Unfortunately, it has not been known that an estimator \(\phi_{l,n,a}\) is strongly consistent with \(\phi_{l}(t_0)\). Thus, we will investigate numerically its asymptotic property of the resulting estimator of their optimal spare part delivery schedule.

5. Simulation Experiments

Of our interest in this section is the investigation of asymptotic properties of estimators proposed in Sections 4. Suppose that the lifetime obeys the Weibull distribution:

\[
F(t) = 1 - e^{\frac{t^\gamma}{\theta^\gamma}}
\]

(50)

with the shape parameter \(\gamma = 2.0\) and the scale parameter \(\theta = 1.0\). In this situation, the MTTF (mean time to failure) is given by \(\mu = 0.8862\), and the failure rate \(\eta(t)\) is strictly increasing in \(t\), i.e., the manufacturing machine tends to fail as the time elapses. The other model parameters are fixed as \(c_1 = 0.9\) ($), \(c_2 = 0.5\) ($), \(\alpha = 0.8\), \(q = 1.2\) ($). Figure 2 illustrates an example to determine the optimal spare delivery schedule under the Weibull lifetime assumption. Here we concern the determination problem of the optimal spare part delivery schedule under the expected total discounted profit over an infinite time horizon. For the case under the long-run profit per unit time in the steady state, see Dohi et al. (2000). In this example, since we have \(\xi_a = -0.833333\) and \(\eta_a = -0.653917\), the optimal point with the maximum tangent slope and its corresponding modified TTT transform are given by \(1 - \exp(-\alpha x_0) = 0.53606\) and \(\phi_a(t_0) = 0.904762\), respectively. Then, the optimal spare delivery schedule is calculated by \(t_0^* = 0.96\) (days) with \(P_d(0.96) = 3.55968\) ($).

Next, let us consider an estimation problem. Generate 100 random numbers as the lifetime data sampled from the Weibull distribution given in Eq.(50). In Figure 3, we present an estimation example of the optimal spare part delivery schedule under the expected total discounted profit over an infinite time horizon. In this case, the optimal point can be estimated by \(1 - \exp(-\alpha x_0) = 0.545943\) with the maximum tangent slope from \(S_a = (-0.833333, -0.662212)\). From this result, we estimate the optimal spare part delivery \(x_i = x_{61} = 0.986942\) (days) and the maximum expected profit \(P_d(t_0^*) = 3.57586\) ($). We examine the asymptotic behavior of the estimate based on the modified TTT plot. Figures 4 and 5 depict the asymptotic results for the optimal delivery schedule and its expected total discounted profit, respectively. From Figure 4, it can be seen that the estimate of the optimal spare delivery time converges to the real optimal almost accurately when more than 30 lifetime data are observed. Also, in Figure 5, we observe that the expected total discounted profit can be estimated accurately with more than 30 data and the developed estimate of the optimal spare delivery time.

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**Figure 2. Determination of the Optimal Spare Part Delivery Schedule:** \(\gamma = 2.0\), \(\theta = 1.0\), \(c_1 = 0.9\) ($), \(c_2 = 0.5\) ($), \(\alpha = 0.8\), \(q = 1.2\) ($).

**Figure 3. Estimation of the Optimal Spare Part Delivery Schedule:** \(c_1 = 0.9\) ($), \(c_2 = 0.5\) ($), \(\alpha = 0.8\), \(q = 1.2\) ($).
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