Obnoxious Facility Location Problem with Forbidden Regions

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Abstract

In this paper, we consider an optimal location problem for an obnoxious facility such as disposal center. We formulate this problem as a single facility minimax location problem on the plane under the presence of forbidden regions where the facility is not permitted. The solution for a minimax location problem without forbidden regions is known to be a center of minimum covering circle, but we must consider the case where the solution is not feasible. First we seek the optimal feasible solution with Euclidean distance, secondary we consider it with rectilinear distance. In order to solve these problems, we derive some useful properties and fully utilizing them, we propose efficient algorithms to find the optimal location.

Keywords: Location problem; Obnoxious facility; Non-linear optimization

1. Introduction

One of major problems concerning with the location of a single facility on the plane is to determine the location that is the closest to even the farthest distant user. This problem is known as the minimax location problem, i.e., the problem to find the optimal facility location which minimizes the maximum of distances between the facility and all demand points.

The minimax criterion is generally accepted as a valid criterion to find the optimal location of a facility which offers public services. For example, fire stations, hospitals, schools, airports, disposal center, etc. In the other criterion such as minisum case, residents in a remote place will be sacrificed for the benefit of the many. In the basic minimax model, there is no constraint with the undesirable effects, i.e., whole plane is feasible region. But actually, there will be noise and pollution associated with the facility. Dealing with such an obnoxious facility, we assume that there exists some area in the plane where facility location is not permitted. We call such an area as forbidden region. Note that there is no restriction to travel through them.

First, under the presence of forbidden regions, we consider the optimal facility location minimizing the maximum distances between the facility and all demand points with Euclidean distance. Secondary, we consider the same problem with rectilinear distance.

2. Minimax Problem with Forbidden Regions

In this section, we formulate a minimax location problem in the presence of forbidden regions with general distance function. Suppose there exist \( l \) demand points. We assume that all of them have forbidden regions, where the facility is not permitted. Additionally, we introduce \( n-l \) forbidden regions which are independent from demand points. The forbidden regions such as parks are approximated by the union of these demand independent forbidden regions.

We use following notations.

\( x \): location of the facility
\( a_i \): location of demand points \( (i = 1, \ldots, l) \)
\( a_i \): center of forbidden regions independent from demand points \( (i = l+1, \ldots, n) \)
\( F_i \): forbidden region centered at \( a_i \)
\( r_i \): radius of \( F_i \)
\( B_i \): boundary of \( F_i \)
\( d(p, q) \): distance between \( p \) and \( q \)

Then the forbidden region \( F_i \) and its boundary \( B_i \) are defined by

\[ F_i = \{ p \mid d(p, a_i) < r_i \}, \quad B_i = \{ p \mid d(p, a_i) = r_i \}, \quad i = 1, \ldots, n. \]

Note that the boundary \( B_i \) is a feasible region since \( F_i \) is a open set. As the radii are arbitrary nonnegative numbers, in the case that a certain demand point does not
accompany a forbidden region, we regard it as the case that the radius is zero.

We define \( F \) by
\[
F = \bigcup_{i=1}^{n} F_i.
\]
So the feasible region is \( R \setminus F \).

Our minimax problem \( P_1 \) with forbidden regions is to determine \( x \) as follows.

\[
P_1 : \text{minimize } \max_{a_i \in P} d(x, a_i)
\]
subject to \( x \in \mathbb{R}^2, F \)

Here, we define the farthest Voronoi region \( FV(a_i) \).

It is defined by
\[
FV(a_i) = \{ p | d(p, a_i) \geq d(p, a_j), a_j \in P, \{ a_j \} \}
\]
where \( P \) is the set of all demand points. A boundary of the farthest Voronoi region is called a farthest Voronoi edge, and an intersection of more than two edges is called a farthest Voronoi point. Only the points determining a convex hull have the farthest Voronoi regions. It is known that the center of the minimum covering circle with Euclidean distance is equal to one of the farthest Voronoi points.

3. Solution with Euclidean Distance

If there is no forbidden region, it is well known that the solution for a minimax location problem with Euclidean distance becomes the center of the minimum covering circle. Such a solution can be found by the order \( O(l \log l) \) (Elzinga and Hearn, 1972; Shamos and Hoey, 1975).

Let \( C \) and \( O \) denote the minimum covering circle and its center respectively. In the case where the forbidden region \( F \) includes \( O \), next property holds.

**Property 1.** If \( O \in F \) then the optimal location \( x^* \) exists on the boundary of \( S \), which is constructed by the following procedure: (1) \( S \leftarrow \emptyset \); (2) for all \( i \), if \( F_i \cap O \neq \emptyset \) or \( F_i \cap S \neq \emptyset \) then \( S \leftarrow F_i \cup S \).

**Outline of Proof.** Assume that \( O \in S \) and a point \( X \) in the feasible region is a candidate point of the optimal solution. On the line segment \( OX \), we can find the first feasible point \( Y \) scanning from \( O \) to the direction \( X \). Draw minimum circles centered with \( O, X, Y \) including all demand points. The three circles have at least one common demand point on them. Comparing the radii of the circles, \( Y \) is the better solution than \( X \). Therefore, the optimal solution exists on the boundary \( B \) of \( S \).

From now we call \( S \) as a connected forbidden region. Then the property is restated that the optimal solution exists on the boundary \( B \) of the connected forbidden region. An example of \( B \) is illustrated with thick curves in Figure 1. Note that it is not always true that \( x^* \) is on the boundary \( B_i \) of \( F_i \) which contains \( O \), since a point on \( B_i \) may be covered with the other forbidden regions.

The important point to note is that the nearest point \( A \) from \( O \) on the boundary \( B \) is not always the optimal solution, since \( A \) is not always the point minimizing the distance to the farthest demand point. Such a case is illustrated in Figure 2. Suppose the nearest point \( A \) is on \( B_i \), and it is included by \( FV(a_j) \). The radius of the minimum covering circle is determined by the distance between \( a_j \) and the solution point for \( P_1 \).

**Property 2.** The candidate solutions are the extreme points of \( B_i \), and the intersections of \( B_i \) and the farthest Voronoi edges.

**Outline of Proof.** It is obvious from the convexity of circles that the the extreme points of \( B_i \) are the candidate solutions.
ing \( FV(a_j) \) and \( FV(a_k) \). In the case where \( B_i \), intersects \( E_{jk} \), the intersection point \( X \) is at the same distance from \( a_j \) and \( a_k \). It means that the radius of the minimum covering circle centered with \( X \) is \( d(a_j, X) = d(a_k, X) \). Assume a point \( Y \neq X \) on \( B_i \) is in \( FV(a_j) \), then \( d(a_k, Y) \) is greater than \( d(a_j, X) \) from the definition of the farthest Voronoi region. \( Y \) increases the radius of the minimum covering circle, so the \( X \) is the better solution than \( Y \).

Therefore the solution procedure for \( P1 \) is described as follows.

Step 1. Find the minimum covering circle without forbidden regions.

Step 2. Construct the connected forbidden region and its boundary \( B \).

Step 3. Construct the convex hull and the farthest Voronoi diagram.

Step 4. For each \( B_i \), store the extreme points and the intersections with the farthest Voronoi edges as the candidate solutions. At the same time, store the distance to the farthest demand points from the candidate solutions.

Step 5. Among the enumerated candidate solutions, find the optimal solution minimizing the distance to the farthest demand point.

4. Solution with Rectilinear Distance

Rectilinear distance is defined by

\[
d(p, q) = |p_1 - q_1| + |p_2 - q_2|
\]

where \((p_1, p_2)\) and \((q_1, q_2)\) are the coordinate of \( p, q \).

In general, the set of points at the same distance from a certain point \( p \) is called a circle centered with \( p \). It is noteworthy that the “circle” with rectilinear distance becomes a diamond visually.

If there is no forbidden region, as the case of Euclidean distance, the solution for a minimax location problem with rectilinear distance becomes the center of the minimum covering circle. The difference is that the center of the minimum covering circle is not determined uniquely, i.e., the set of the center becomes a line segment.

Let \( Q \) denotes the set of the center \( O \) of the minimum covering circle. If \( Q, F \) is not an empty set, then the solution is an arbitrary point on \( Q, F \). In the case where the forbidden region includes \( Q \), next property holds.

Property 3. If \( Q, F = \emptyset \) then the optimal location \( \chi^* \) exists on the boundary \( B \) of the connected forbidden region \( S \).

This property is proved in the similar way as the case of Euclidean distance. An example of \( B \) is illustrated with thick lines in Figure 3.

The minimum covering circle with Euclidean distance can be found by the order \( O(l \log l) \). But in the case of rectilinear distance, the order is reduced to \( O(l) \) as follows.

The diameter of minimum covering circle with rectilinear distance is determined by the farthest pair of demand points. Generally the set \( Q \) of the center \( O \) becomes a line segment with slope of \( -1 \) or 1. For labeling some points, rotate all points temporarily by multiplying the coordinates with a matrix \( A \) where

\[
A = \begin{pmatrix}
\cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\
\sin \frac{\pi}{4} & \cos \frac{\pi}{4}
\end{pmatrix}.
\]

After this transformation, find the points which have the minimum and the maximum coordinate values on each coordinate. Let \( a^1 \) and \( a^2 \) denote the points which have the minimum and the maximum values respectively with \( x \)-coordinate. We also define \( a^3 \) and \( a^4 \) as the minimum and the maximum points with \( y \)-coordinate. If some points have the same value, an arbitrary one of them is chosen as a representative point. After this labeling, restore the location of all points.

Without loss of generality, we assume \( d(a^1, a^3) \geq d(a^3, a^4) \). This means that \( a^1 \) and \( a^2 \) are the farthest pair. Then \( Q \) is a part of the perpendicular bisector of \( a^1 \) and \( a^2 \) with rectilinear distance. The extreme points of \( Q = \{(x, y)\} \) can be obtained as the solution of the following simultaneous equations, setting \( k \) to 3 or 4. \( a^1_x \) and \( a^1_y \) mean the coordinate values of \( a^1 \).
Thus the set \( Q \) of the center of the minimum covering circle can be found by the order of \( O(l) \). Let \( Q_1 \) and \( Q_2 \) denote the extreme points of line segment \( Q \).

Unlike the case of the Euclidean distance, the increment of the radius of the minimum covering circle is equal to the distance between \( Q \) and the nearest point on \( B \). So the optimal solution with rectilinear distance is the point or the line segment on \( B \) minimizing the distance from \( Q \), i.e., the nearest feasible region from \( Q \). So the next property holds.

**Property 4.** The candidate solutions are the intersections of \( B_i \) and \( B_j \), and the intersections of \( B_i \) and the farthest Voronoi edges, where \( B_i \cap B \neq \emptyset \). \( B_j \cap B \neq \emptyset \).

Let \( I_{ij} \) denote the intersection of \( B_i \) and \( B_j \). Using the method of computational geometry, we can enumerate the intersections by the order of \( O(l \log l) \) (Preparata and Shamos, 1985; Imai and Asano, 1983).

The farthest Voronoi region \( FV(a_i) \) exists only when \( a_i \) is one of the points which has an intersection with the minimum covering circle. Such points are \( a_1, \ldots, a_4 \), labeled in the above. But in the case of degeneracy, not all of them have the corresponding regions. Namely the farthest Voronoi region with rectilinear distance have 4 regions separated by 5 edges at most. The edges are parallel to the lines represented \( y = x \) or \( y = -x \), or parallel to \( x \)-axes or \( y \)-axes.

Now we investigate the relationship between farthest Voronoi diagram and minimum covering circle with rectilinear distance.

**Property 5.** The set of the center of minimum covering circle is exactly equal to one of the farthest Voronoi edges. It is the edge which contains the middle point of the farthest pair \( a^1 \) and \( a^2 \), and separating the region \( FV(a^1) \) and \( FV(a^2) \).

**Outline of Proof.** Suppose \( C \) is the minimum covering circle passing through \( a^1, a^2 \) and \( a^3 \). Then the Voronoi point is the center of \( C \).

Let \( d(Q, x) \) represent the minimum distance from \( x \) to a point on \( Q \). Then the solution procedure for \( P_1 \) with rectilinear distance is described as follows.

Step 1. Enumerate the intersection \( I \) of \( B_i \) and \( B_j \).
Step 2. Enumerate the intersection \( I_{ij} \) of \( B_i \) and the farthest Voronoi edges \( E_k \).
Step 3. Choose \( I_{ij} \) minimizing \( d(Q, I_{ij}) \).

This algorithm check all the possibility, but what we need for the candidate solutions are the enumerate the extreme points of \( B_i \), so can reduce the enumeration of \( I_{ij} \).

by constructing the connected forbidden region. So the improved version of the algorithm becomes as follows.

Step 1. Find the \( a^1, \ldots, a^l \) and \( Q \).
Step 2. Construct the connected forbidden region and its boundary \( B \).
Step 3. For each \( B_i \), save the extreme points and the intersections with the farthest Voronoi edges as the candidate solutions. At the same time, save the distance to the farthest demand points from the candidate solutions.
Step 4. Among the enumerated candidate solutions, find the optimal solution minimizing the distance to the farthest demand point.

5. Conclusion and Future Extension

In order to determine the reasonable location for obnoxious facility, we introduced a concept of forbidden regions, which approximate any shape of restrictions in the real world. We formulated the problem and showed that the optimal location is on the boundary of connected forbidden regions which includes the center of the minimum covering circle, and proposed a solution procedure to find the solution. In this paper, we introduced two different functions, i.e., Euclidian distance and rectilinear distance, for estimating the distance between the demand points and the facility. They will be valid model for noise and traffic in urban area respectively.

Our further research will be on the same minimax type model with conflict criteria. We are interested in the optimal facility location maximizing the minimum degree of satisfaction of the residents, and the satisfaction level monotonically decreases as the facility approaches to them due to the undesirable elements of the facility such as noise and pollution. That will be a model for considering minimax criterion and maximin criterion simultaneously.

**References**


