An Application of Particle Swarm Optimization to Bilevel Facility Location Problems with Quality of Facilities

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Abstract

This paper proposes a new model of bilevel facility location problems (BFLPs). Drezner considered a BFLP on a plane, which is an optimal location problem for the situations that there are two types of decision makers (DMs), an upper DM and a lower DM, and they locate some competitive facilities, e.g. shops and stores. An optimal location for BFLP is generally represented as Stackelberg solution. In Drezner’s BFLPs, customers choose their using facility by considering only the distance between them and the facilities. This paper considers a BFLP with quality of facilities, e.g. scale of facilities, good service, and variety of goods. In order to solve the formulated BFLP efficiently, a solving method for the BFLP based upon particle swarm optimization (PSO) methods is proposed. The efficiency of the proposed solving method is shown by applying to some examples of the BFLPs.

Keywords: Facility location; Competitiveness; Quality of facilities; Stackelberg solution; Particle swarm optimization

1. Introduction

A competitive facility location problem (CFLP) is one of optimal location problem for competitive facilities, e.g. shops, supermarkets, etc. Hotelling (1929) considered CFLPs under the conditions that (i) customers are uniformly distributed on a line segment, (ii) each of decision makers (DMs) can locate and move her/his own facility at any times, and (iii) all customers only use the nearest facility. In Hotelling’s CFLPs, an optimal location for all facilities is represented as Nash solution. Wendell and McKelvey (1981) proposed CFLPs that there exist customers on a finite number of points, called demand points (DPs), and the DMs locate their facilities on a network whose nodes are DPs.

Based upon the CFLPs of Wendell and McKelvey, Hakimi (1983) considered bilevel facility location problems (BFLPs) under the conditions that (i) there are two types of DMs, which are an upper DM and a lower DM, (ii) the upper DM first decide the location of her/his own facilities, and secondly the lower DM locates her/his own facilities for the location of the upper DM, and (iii) both of the DMs cannot move their facilities. In Hakimi’s BFLP, an optimal location for all facilities is represented as Stackelberg solution. Hakimi showed that the BFLPs are NP-hard. For details of the BFLPs on a network, the reader can refer the study of Miller et al. (1996). Drezner (1982) considered BFLPs on a plane that there exist DPs, and proposed an effective solving method with using linear programming method and bisectional method for the cases that each of the DMs locates one facility.

In the above BFLPs, it is assumed that customers only use the nearest facility. However, if there is a difference in quality between the facilities, e.g. scale of facilities, good service and variety of goods, customers do not always use the nearest facility. Karkazis (1989) considered BFLPs on a network by considering the quality of the facilities. Although he proposed a solving method to find an optimal solution of the lower DM, the BFLPs is also NP-hard.

In this paper, we propose a new BFLP on a plane by considering the quality of the facilities based on an attractive power function suggested by Huff (1964). We formulate BFLPs as reward maximization problems by considering a building cost of facilities. In order to solve the formulated BFLPs efficiently, we propose a solving method for the BFLPs based upon particle swarm optimization (PSO) methods, proposed by Kennedy and Eberhart (1995). The efficiency of the proposed solving method is shown by applying to some examples of the BFLPs.

The remaining structure of this article is organized as follows: In Section 2, we introduce Huff’s attractive power function to Drezner’s BFLPs on a plane, and formulate BFLPs as reward maximization problems for the upper DM. In order to solve the formulated BFLPs efficiently, we propose an efficient solving method based on PSO

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methods in Section 3. In Section 4, we show the efficiency of the solving method by applying the method to several examples of the BFLPs. Finally, in Section 5, concluding comments and future extensions are summarized.

2. Formulation of BFLP with Huff's Attractive Power Function

In this section, we formulate a BFLP with quality of facilities on a plane. First, we formulate the BFLP proposed by Drezner (1982). In the BFLP, it is assumed that all customers only exist on a finite number of DPs in plane $R^2$. For convenience sake, one DP is dealt with a customer by regarding all customers on the same DP as a group.

There are $n$ DPs on a plane $R^2$, and let $D = \{1, ..., n\}$ be a set of indices of the DPs. For DP $j \in D$, its site is denoted by $u_j \in R^2$. In the plane $R^2$, each of the upper and the lower DMs locates her/his one facility, called the upper and the lower facility, respectively. Let $x$ and $y \in R^2$ be the site of the upper and the lower facility, respectively.

It is assumed that customers only use the nearest facility and if the distances of these facilities are the same, they only use the upper facility. Then, we use the following binary variable in order to represent whether DP $j$ uses the upper facility:

$$\phi_j(x, y) = \begin{cases} 1, & \text{if } \|x-u_j\| \leq \|x_j-u_j\| \\ 0, & \text{if } \|x-u_j\| > \|x_j-u_j\| \end{cases} \tag{1}$$

Here $\|\cdot\|$ is the Euclidian norm in $R^2$.

Let $w_j > 0$ denote a buying power (BP) of DP $j$. Each of the facilities can obtain the BPs from the customers using its facility. The objective functions for the upper and the lower DMs are the sum of obtaining BPs and represented as follows:

$$f_1(x, y) = \sum_{j=1}^{n} \phi_j(x, y) \cdot w_j, \tag{2}$$

$$f_2(x, y) = \sum_{j=1}^{n} \{1 - \phi_j(x, y)\} \cdot w_j. \tag{3}$$

Then, the Drezner's BFLP is formulated as follows:

$$\begin{align*}
\text{maximize} & \quad f_1(x, y) \\
\text{where} & \quad y \text { solves } \\
\text{maximize} & \quad f_2(x, y) \\
\text{subject to} & \quad x \in R^2, y \in R^2 \\
\end{align*} \tag{4}$$

Next, we formulate a BFLP with the quality of the facilities by extending BFLP (4). Let $L$ be the maximal level of the quality of the facilities that the upper and the lower DM can locate, and let $l_1$ and $l_2 \in \{1, ..., L\}$ be the levels of the quality of the upper and lower facilities, respectively. Then, we use the following attractive power function suggested by Huff (1964) for representing the attractiveness of the facility whose site and level is $z \in R^2$ and $l \in \{1, ..., L\}$ for DP $j$:

$$a_j(z, l) = \begin{cases} \frac{q(l)}{\|z-u_j\|}, & \text{if } \|z-u_j\| \geq \varepsilon, \\ \frac{q(l)}{\varepsilon^2}, & \text{if } \|z-u_j\| < \varepsilon, \end{cases} \tag{5}$$

where $q(l)$ is a quality of the facility such that $0 < q(1) < \cdots < q(L) < \infty$, and $\varepsilon$ is an upper limit of the distance that customers can move without any trouble.

We assume that customers only use the upper facility if its attractiveness is more than attractiveness of the lower facility; otherwise they only use the lower facility. Then, we use the following binary variable in order to represent whether DP $i$ uses the upper facility:

$$\phi_j(x, l_1, y, l_2) = \begin{cases} 1, & \text{if } a_j(x, l_1) \geq a_j(y, l_2), \\ 0, & \text{if } a_j(x, l_1) < a_j(y, l_2). \end{cases} \tag{6}$$

Let $c_1(l)$ and $c_2(l)$ denote the building costs of the upper and the lower facilities whose levels are $l$, respectively. Here, they are satisfied that $0 < c_1(l) < \cdots < c_1(L) < \infty$ and $0 < c_2(l) < \cdots < c_2(L) < \infty$. Then, we represent objective functions for the upper and the lower DM as follows:

$$f_1(x, l_1, y, l_2) = \alpha \cdot \sum_{j=1}^{n} \phi_j(x, l_1, y, l_2) \cdot w_j - c_1(l_1), \tag{7}$$

$$f_2(x, l_1, y, l_2) = \alpha \cdot \sum_{j=1}^{n} \{1 - \phi_j(x, l_1, y, l_2)\} \cdot w_j - c_2(l_2). \tag{8}$$

where $\alpha$ is a positive number. Therefore, a BFLP with the quality of the facilities is formulated as follows:

$$\begin{align*}
\text{maximize} & \quad f_1(x, l_1, y, l_2) \\
\text{where} & \quad y \text{ and } l_2 \text{ solve } \\
\text{maximize} & \quad f_2(x, l_1, y, l_2) \\
\text{subject to} & \quad x \in R^2, l_1 \in \{1, ..., L\}, y \in R^2, l_2 \in \{1, ..., L\} \\
\end{align*} \tag{9}$$

For a solution of the upper DM, the solution of the lower DM which maximizes the objective function value of the lower DM is called a rational response. Then, a pair of the solution of the upper DM and its rational response which maximizes the objective function value of the upper DM is an optimal solution of problem (9) and called a Stackelberg solution. In the next section, we propose an efficient solving method to find a Stackelberg solution.
3. Solving Method for BFLP

In Section 3.1, we propose the method for finding a rational response of the lower DM for each solution of the upper DM. In Section 3.2, we propose the solving method for finding a Stackelberg solution.

3.1. Method for Finding a Rational Response of the Lower DM

First, we solve the following problem for each \((x, l_1)\):

\[
\begin{align*}
\text{maximize} & \quad f_2(x, l_1, y, l_2) \\
\text{subject to} & \quad y \in \mathbb{R}^2, l_2 \in \{1, \ldots, L\}.
\end{align*}
\]  

(10)

Since problem (10) is a nonconvex nonlinear programming problem, it is difficult to solve the problem directly. Then, we reduce the problem to a combinational optimization problem.

From equation (6), the new facility can obtain BPs from all DPs in a subset \(D \subseteq D\) if there exists \((\bar{y}, \bar{l}_2)\) such that

\[
a_j(x, l_1) \leq a_j(y, l_2), \quad \forall j \in \bar{D}.
\]  

(11)

In order to find \(\bar{y}\) satisfying equation (11) for a given \(\bar{l}_2 \in \{1, \ldots, L\}\), we solve the following problem with an auxiliary variable \(\gamma\):

\[
\begin{align*}
\text{minimize} & \quad \gamma \\
\text{subject to} & \quad a_j(x, l_1) \cdot \gamma \leq a_j(y, \bar{l}_2), \quad \forall j \in D, \\
& \quad y \in \mathbb{R}^2, \gamma \geq 0.
\end{align*}
\]  

(12)

Then, the following theorem is shown by Uno and Katagiri (2007).

**Theorem 1.** For a given subset \(D \subseteq D\) and level \(\bar{l}_2 \in \{1, \ldots, L\}\), let \((\gamma^\bar{D}, y^\bar{D})\) be the solution of problem (12). Then, if \(\gamma^\bar{D} < 1\), the lower DM can obtain BPs from all DPs in \(\bar{D}\) by locating the lower facility whose site and level are \(y^\bar{D}\) and \(\bar{l}_2\), respectively.

From Theorem 1, we can reformulate problem (10) to the following combinational optimization problem:

\[
\begin{align*}
\text{maximize} & \quad f_2(x, l_1, y^\bar{D}, \bar{l}_2) \\
\text{subject to} & \quad a_j(x, l_1) \cdot \gamma^\bar{D} \leq a_j(y^\bar{D}, \bar{l}_2), \\
& \quad \bar{l}_2 \in \{1, \ldots, L\}, \bar{D} \subseteq D.
\end{align*}
\]  

(13)

In order to solve problem (13), we need to solve problem (12) for all pairs of the subset of \(D\) and the level of the lower facility. We propose to reduce problem (13) by decreasing the number of the times to solve problem (12).

Let \(D_3\) be the class of sets which include at most three DPs. Then, the following theorem is useful to decrease the number of the times, proved by Uno and Katagiri (2007).

**Theorem 2.** An optimal solution of problem (13) can be found by solving problem (12) for all sets in \(D_3\).

Note that \(y^\bar{D}\) depends only on subset \(\bar{D}\), that is, independent of level \(\bar{l}_2\). Then, if there exists \(\bar{l}_2\) such that \(\gamma^\bar{D} < 1\), an optimal level \(\bar{l}_2\) for \(y^\bar{D}\) is the minimal level of all the levels because of building costs. We denote the above minimal level by \(\bar{l}^\bar{D}_2\), where if there is no \(\bar{l}_2\) such that \(\gamma^\bar{D} < 1\), we set \(\bar{l}^\bar{D}_2 = L\). We thus reduce problem (13) to the following combinational optimization problem:

\[
\begin{align*}
\text{maximize} & \quad f_2(x, l_1, y^\bar{D}, \bar{l}_2^\bar{D}) \\
\text{subject to} & \quad a_j(x, l_1) \cdot \gamma^\bar{D} \leq a_j(y^\bar{D}, \bar{l}_2^\bar{D}), \\
& \quad \bar{l}_2^\bar{D} \in D_3.
\end{align*}
\]  

(14)

We can find an optimal solution of problem (14) by solving problem (12) at \(n + \alpha C_2 + \alpha C_3\) times.

3.2 Solving Method for Stackelberg Solution of BFLP

We first introduce an outline of the solving method for BFLP (9), and then we introduce a PSO method used in the proposed solving method. Let \((x^*, l_1^*, y^*, l_2^*)\) be the temporary Stackelberg solution given in the following algorithm, and \(f_1^*\) be the objective value of its solution.

**Algorithm 1: Outline of the solving method for BFLP (9)**

Step 1: Set \(\hat{l} = 1\).

Step 2: Solve the following BFLP by using a PSO method:

\[
\begin{align*}
\text{maximize} & \quad f_1(x, \hat{l}, y, l) \\
\text{where} & \quad y, l \text{ solves } \\
\text{maximize} & \quad f_2(x, \hat{l}, y, l) \\
\text{subject to} & \quad x \in \mathbb{R}^2, y \in \mathbb{R}^2, l \in \{1, \ldots, L\}.
\end{align*}
\]  

(15)

Step 3: Let \(x_1^\hat{l}\) denote an optimal solution of BFLP (15) for \(l_1 = \hat{l}\), and \((y^\hat{l}, \bar{l}_2^\hat{l})\) be the rational response for the solution of the upper DM \((x, l_1) = (x_1^\hat{l}, \hat{l})\).

If \(f_1^* < f_1(x_1^\hat{l}, \hat{l}, y^\hat{l}, \bar{l}_2^\hat{l})\), then set \((x^*, l_1^*, y^*, l_2^*) \leftarrow (x_1^\hat{l}, \hat{l}, y^\hat{l}, \bar{l}_2^\hat{l})\) and
from equation (5), equation (6) can be, $BFLP^*=BFLP(9)$ is equivalent to Drezner’s $BFLP(4)$. Otherwise, set $\hat{l}_{t+1} = \hat{l}_{t} + 1$ and return to Step 2.

The PSO method is based on the social behavior that a population of individuals adapts to its environment by returning to promising regions that were previously discovered (Kennedy and Spears, 1998). This adaptation to the environment is a stochastic process that depends both on the memory of each individual, called particle, and on the knowledge gained by the population, called swarm.

In the numerical implementation of this simplified social model, each particle has three attributes: the position vector in the search space, the velocity vector and the best position in its track, and the best position of the swarm. The process can be outlined as follows:

Algorithm 2: Outline of the PSO method

Step 1: Generate the initial swarm involving $N$ particles at random.

Step 2: Calculate the new velocity vector for each particle, based on its attributes.

Step 3: Calculate the new position of each particle from the current position and its new velocity vector.

Step 4: If the termination condition is satisfied, then stop. Otherwise, go to Step 2.

Let $x_i^t$ denote the position of the $i$-th particle at time $t$, and $v_i^t$ denote the velocity vector of the $i$-th particle at time $t$. Then, $v_i^{t+1}$, which is the velocity vector of the $i$-th particle at time $t+1$, is calculated by the following scheme (Shi and Eberhart, 1998):

$$v_i^{t+1} := \omega v_i^t + \beta_1 R_i^1 (p_i^t - x_i^t) + \beta_2 R_i^2 (p_g^t - x_i^t) \quad (16)$$

In (16), $R_i^1$ and $R_i^2$ are random numbers between 0 and 1, $p_i^t$ is the best position of the $i$-th particle in its track and $p_g^t$ is the best position of the swarm. There are three parameters; the inertia of the particle $\omega^t$, and two trust parameters $\beta_1, \beta_2$.

Then, the new position of the $i$-th particle at time $t+1$ is calculated as follows:

$$x_i^{t+1} := x_i^t + v_i^{t+1} \quad (17)$$

The $i$-th particle calculates the next search direction vector $\vec{v}_i^{t+1}$ by equation (16) in consideration of the current search direction vector $\vec{v}_i^t$, the direction vector going from the current search position $x_i^t$ to the best position in its track $p_i^t$ and the direction vector going from the current search position $x_g^t$ to the best position of the swarm $p_g^t$, moves from the current position $x_i^t$ to the next search position $x_i^{t+1}$ calculated by (17). The parameter $\omega^t$ controls the amount of the move to search globally in early stage and to search locally by decreasing $\omega^t$ gradually. It is defined by follows:

$$\omega^t := \omega^0 - \frac{t(\omega^0 - \omega^{T_{\text{max}}})}{0.75 \omega^{T_{\text{max}}}} \quad (18)$$

where $T_{\text{max}}$ is the number of maximum iteration times, $\omega^0$ is an initial value at the time iteration, and $\omega^{T_{\text{max}}}$ is the last value.

The searching procedure of PSO is shown in Figure 1 and 2. Comparing the evaluation value of a particle after movement, denoted by $f_i(x_i^{t+1})$, with that of the best position in its track, denoted by $f_i(p_i^{t+1})$, if $f_i(x_i^{t+1})$ is better than $f_i(p_i^{t+1})$, then the best position in its track is updated as $p_i^{t+1} := x_i^{t+1}$. Furthermore, if $f_i(p_i^{t+1})$ is better than $f_i(p_g^{t+1})$, then the best position in the swarm is updated as $p_g^{t+1} := p_i^{t+1}$. In this way, a particle gets information of the best position of new oneself and the swarm, and moves according to (16) again, and searches newly.

Such a PSO technique includes two problems. One is that particles concentrate on the best search position of the swarm and they cannot easily escape from the local optimal solution since the move direction vector $v_i^{t+1}$ calculated by (16) always includes the direction vector to the best search position of the swarm. Another is that a particle after move is not always feasible for problems with constraints. Matsui et al. (2006) proposed a PSO method improving for the above two problems.

We apply the PSO method improved by Matsui et al. (2006) for solving problem (15). First we consider the generation of the initial swarm in Step 1 of Algorithm 2. The following theorem is useful for the generation.

**Theorem 3.** If the lower DM locates her/his facility that $l_1 = l_2$, BFLP (9) is equivalent to Drezner’s BFLP (4).

**Proof.** If $l_1 = l_2$, from equation (5), equation (6) can be transformed as follows:

$$\phi(x_1, l_1, x_2, l_2) = \begin{cases} 
1, & \text{if } \|x_1 - v_i^1\| \leq \|x_2 - v_i^1\|, \\
0, & \text{if } \|x_1 - v_i^1\| > \|x_2 - v_i^1\|, \quad i = 1, \ldots, n.
\end{cases} \quad (19)$$
Then, equation (16) is equivalent to equation (1). This means that BFLP (9) is equivalent to Drezner’s BFLP (4).

Q.E.D.

For BFLP (4), Drezner (1982) proposed an efficient solving method to find optimal sites with a solving method based upon linear programming and binary method. The complexity of his method is \( O(n^4 \log n) \).

The location of the upper facility found by Drezner’s solving method is an optimal solution for BFLP (15) if the level of the lower facility for its rational response is equal to the level of the upper facility. This means that the location of the upper facility found by Drezner’s solving method is a candidate of optimal solutions of problem (15) for any \( l_1 \). Therefore, we use the location as an initial swarm in Step 1 for any \( l_1 \).

4. Numerical Examples and Experiments

In this section, we apply the proposed solving method to several examples of the BFLPs. In these examples, the numbers of DPs \( n = 20, 40, 60 \). For DP \( j \in D \), its site is given that \( v_j \in [0,100] \times [0,100] \) randomly, and its BP is given that \( w_j \in \{1, \ldots, 100\} \) randomly, whose sums of BPs with all DPs are equal to 50n. For the level of the upper and lower facility, we set \( L = 2 \). The quality and the building cost of the facilities for each level are represented in Table 1. For the other parameters of the BFLPs, we set \( \alpha = 1 \) and \( \varepsilon = 10^{-4} \).

Next, we give the parameters of the PSO method. We set that population size \( N = 30 \), generation \( T_{\text{max}} = 500 \), and the other parameters \( \omega_0 = 1.2 \), \( \omega_{\text{max}} = 0.1 \), and \( \beta_1 = \beta_2 = 2 \).
Moreover, in order to compare with the PSO method, we solve these three BFLPs by a random search. We randomly give the site and level of the upper facility, and then find a rational response of the lower DM by solving problem (14). For each example of the BFLPs, we continue to implement the random search until its computational time is over 10000 seconds.

Results of numerical experiments for these three examples of the BFLPs are given in Table 2. In Table 2, the PSO method is implemented twenty times for each example. The best, mean, and worst values in this table mean the best, the mean, and the worst objective function values of problem (9). The CPU times are the computation times used by DELL Optiplex GX260 (CPU: 2.33 GHz, RAM: 512MByte).

From Table 2, the solving method based upon the PSO method can find better solutions than those of random search by a shorter CPU time for all examples of the BFLPs. This result shows that the solving method is effective for solving BFLP (4).

For all examples of the BFLPs, we find that an optimal level of the upper facility is 2. If we set that the level of the upper facility is 1, an optimal response of the lower facility is that its site is the same as the upper facility and its level is 2. Then, the upper facility cannot obtain BPs from any DPs, so that its objective value is -200.

For the examples with \( n = 20, 40 \), the level of the lower facility in rational responses for the best solution is 1. Although former solving methods, e.g. Drezner’s solving method, cannot solve BFLP in cases that the level of the upper facility is different from that of the lower facility, our solving method can find good solutions at all twenty times.

For the example with \( n = 60 \), the level of the lower facility in rational responses for the best solution is 2. Because the level of the upper facility is equal to that of the lower facility, an optimal site of the upper facility can found by Drezner’s solving method from Theorem 3.

5. Conclusion

In this paper, we have proposed a new BFLP on a plane by considering the quality of the facilities based on an attractive power function suggested by Huff (1964). We have formulated BFLPs as reward maximization problems by considering a building cost of facilities. In order to solve the formulated BFLPs efficiently, we have proposed a solving method for the BFLPs based upon PSO methods proposed by Kennedy and Eberhart (1995). The efficiency of the proposed solving method has been shown by applying to some examples of the BFLPs.

In the future studies, we consider BFLPs for the situation that each of the upper and the lower DMs locates two or more facilities. In order to solve the BFLPs, our solving method is expected to require much computational time. To design more effective solving methods for the BFLPs is an important future study for solving more realistic BFLPs.

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